The Riksbank's communication of macroeconomic uncertainty

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When the forecasts of GDP growth, inflation and the repo rate are presented in Sveriges Riksbank's Monetary Policy Report they are supplemented by associated forecast intervals. In that way the Riksbank emphasizes that forecasts are always uncertain. This article describes in detail how Sveriges Riksbank uses historical forecast errors to calculate these intervals. We propose a number of potential improvements to the current method, including how to handle the repo rate having a lower limit, and how time variation can be introduced in the interval width to automatically adjust for temporary fluctuations in macro-economic uncertainty.

Point forecasts and forecast intervals

The Riksbank has had an inflation target of 2 per cent since 1995. To achieve this target the Executive Board of the Riksbank regularly decides on the repo rate level. Research has shown that monetary policy works with a time lag of 1–2 years before achieving its maximum effect (Christiano, et al., 2005). Consequently, effective monetary policy must be forward-looking and a central part of the Riksbank's work consists of forecasting future economic developments.

In most discussions on forecasts it is implicitly assumed that a forecast is in some sense a qualified guess concerning the future value of a variable. In some cases this *point forecast* is supplemented by an estimate of uncertainty in the forecast, normally presented as a *forecast interval*[2], i.e. a region of values within which the outcome is predicted to land with

[1] This article has been translated from swedish.
[2] It is really more suitable to call these uncertainty intervals, since it is the future outcome that is uncertain, not the forecast. However, it is difficult to find equivalent terminology for what is normally called the forecast distribution (see below) and we will therefore continue to use the term forecast interval.

a given probability, such as 90 per cent, see for example Chatfield (2001, Ch.7) for an introduction. Figure 1 shows how Sveriges Riksbank communicates its forecast of the most important macroeconomic variables in its *Monetary Policy Report*. The solid line is the historical development of the variable and the continuing dashed line is the Riksbank's point forecast 1–12 quarters ahead. The coloured areas illustrate forecast intervals for the respective forecast horizons with three different probabilities: 50, 75 and 90 per cent respectively. The coloured areas are sometimes called forecast bands.

In more general terms one can speak of the *forecast distribution*, i.e. a complete probability distribution for the future value that we are attempting to predict. We make a Bayesian interpretation of the forecast distribution, in which the future value of the variable is unknown at the time of the forecast and can therefore be described with a (subjective) probability distribution. A Bayesian argues as follows: If you do not know the value of a variable (for example CPI inflation in the next quarter) or some other quantity (for example a model parameter) you should describe your lack of knowledge in the form of probabilities. It simply does not matter if the event is random in any deeper sense.[3]

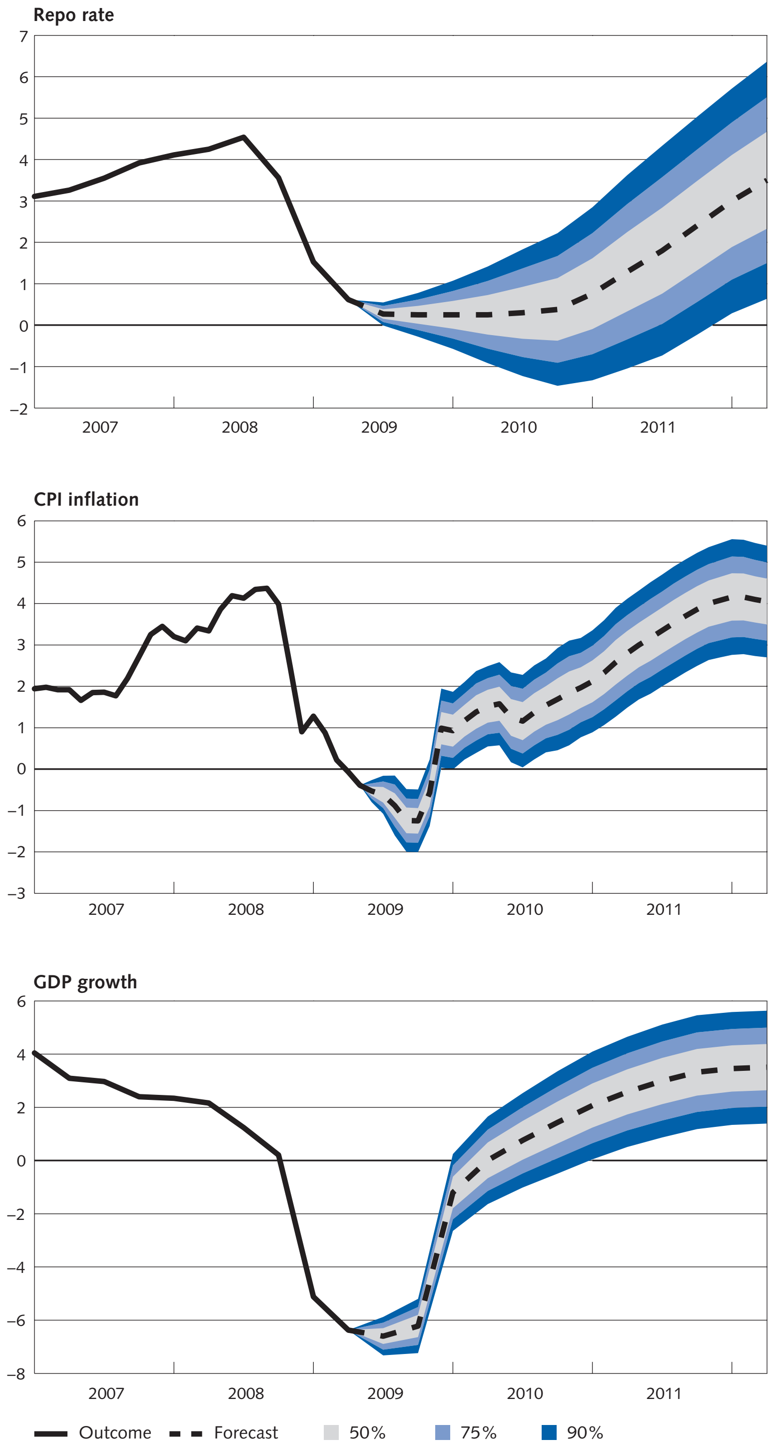
It is natural to describe uncertainty in a decision-making situation (for example before a repo rate decision) in Bayesian terms. The Princeton Professor Chris Sims expressed it as follows: 'Policy discussion at central banks uses the language of Bayesian decision theory' (Sims, 2002).[4] Most of what we describe in this article is, however, applicable even in a non-Bayesian approach, but we opt to make a Bayesian interpretation of the results.

There is an important difference between the Riksbank's point forecasts and the interval around these forecasts. The Riksbank's point forecast for the repo rate conveys its **intentions** for monetary policy, i.e. the policy that the Riksbank intends to implement **if** the economy develops as the Riksbank predicts today. In a similar way, the forecast paths for GDP growth and CPI inflation describe development of these variables given that the Riksbank does not deviate from its proposed repo rate path. On the other hand, the forecast intervals around the point forecasts represent **general** macroeconomic uncertainty (i.e. not only the Riksbank's view

3 The Bayesian pioneer Bruno de Finetti's classical quotation is: "The only relevant thing is uncertainty – the extent of our own knowledge and ignorance. The actual fact of whether or not the events considered are in some sense determined, or known by other people, and so on, is of no consequence" (De Finetti, 1975, page xi).

4 A Bayesian approach is not only preferable in purely conceptual terms but also has many other practical advantages, for example for estimation and model comparison; see for example Bernardo and Smith (1994) for a general discussion or Adolfson et al. (2007b) from a central bank perspective. Three of the Riksbank's most important models, a dynamic general equilibrium model (Adolfson et al., 2008a), a statistical vector autoregressive model (Villani, 2009) and a state-space model with time-varying parameters (Giordani och Villani, 2009), are for example analysed using Bayesian methods exclusively.

Figure 1. Forecast distributions in the Moneatary Policy Report July 2009



of uncertainty). The Riksbank wants to communicate that the future is uncertain and consequently may deviate from today's intentions.[5]

The purpose of this article is to describe in detail how Sveriges Riksbank calculates the forecast bands in Figure 1 on the basis of historical forecast errors. We will give a brief account of alternative approaches, but our ambition is mainly to focus on the method currently used at the Riksbank. After that we describe three potential improvements to the calculations of the Riksbank's forecast interval. The first proposal for improvement is a method that makes it possible to introduce time variation in the forecast interval width so that the intervals can pick up more temporary changes in macroeconomic uncertainty. The second proposal is a simple extension of the Riksbank's present method that takes into account the fact that the repo rate cannot be (too) negative. Finally we point out that the forecast bands in Figure 1 are connected marginal intervals that are to be read forecast horizon by forecast horizon. We then review different ways of designing forecast bands that describe uncertainty in the entire outcome *path* for a variable.

Forecast intervals based on historical forecast errors

In this section we will describe the Riksbank's interval method that is based on the variation in historical forecast errors. There are a number of reasons for this relatively simple idea having gained approval at the Riksbank:

- Using intervals based on historical forecast accuracy is an easily understood way of communicating the uncertainty in the Riksbank's forecasts.
- The Riksbank's point forecast is produced by means of an informal process in which forecasts from structural economic models and statistical forecast models are combined with expert assessments (Hallsten and Tägtström, 2009). The point forecast is therefore well-defined, but cannot be described on the basis of a probability model. Forecast intervals based on historical forecast errors are an attractive alternative when there is no formal probability model that generates the point forecast.
- The correctness of these intervals does not hinge upon on a specific model. The intervals reflect the Riksbank's actual forecast accuracy, regardless of whether the Riksbank's models provide a good description of the economy or not.

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5 However, it is possible to construct forecast bands using intention interpretation, see Svensson and Williams (2007).

A disadvantage of this type of interval is, however, that it is backward-looking and that the interval width is constant over time[6]. In other words the intervals do not automatically become wider in uncertain economic times. In this article we propose a method to introduce time variation into the Riksbank's forecast intervals.

FORECAST INTERVALS FOR GDP GROWTH AND INFLATION

To calculate the forecast intervals for inflation and GDP growth forecasts the Riksbank uses its historical forecast errors for the respective variables. These forecast intervals are illustrated in Figure 1. The forecast errors are defined as a historical forecast for a given quarter minus the outcome for the quarter. This historical spread of forecast errors for each respective forecast horizon is measured by the *root mean squared error* (RMSE)

$$\mathrm{RMSE}(h) = \sqrt{\sum_{t=T}^{T+n-1} (y_{t+h} - y_t^{(h)})^2 / n},$$

where $y_t^{(h)}$ is the h-step forecast made at time t, y_{t+h} is the realised outcome and n is the number of forecast events analysed.

The Riksbank makes forecasts for at least 12 quarters ahead and the spread of the Riksbank's historical forecast errors is calculated for each specific forecast horizon, giving twelve different RMSE values. The relevant RMSE values for CPI inflation and GDP growth that are used in Figure 1 are shown in Table 1.[7, 8]

TABLE 1. RMSE FOR DIFFERENT FORECAST HORIZONS FOR THE RIKSBANK'S FORECASTS OF CPI INFLATION AND GDP GROWTH, 2000–2007.

	Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	Q10	Q11	Q12
CPI inflation	0.30	0.50	0.60	0.65	0.73	0.78	0.81	0.85	0.85	0.85	0.85	0.85
GDP growth	0.44	0.62	0.88	1.00	1.08	1.17	1.23	1.27	1.30	1.30	1.30	1.30

The forecast intervals are calculated on the basis of the assumption that the future forecast errors will follow a normal distribution[9] with an expected value of zero and the same standard deviation as the RMSE of the historical forecast errors. To illustrate the uncertainty in the forecasts as in Figure 1 the distribution of forecast errors for each forecast horizon

6 This is a truth with modification, since the historical forecast errors are continually updated with new outcomes.

7 The forecast of CPI inflation is on a monthly frequency. Here we have only reported the RMSE for the forecast horizons +2, +5, +8, etc. months ahead.

8 The RMSE estimates for long forecast horizons are very uncertain. Consequently, the RMSE for forecast horizons 9–12 quarters have been approximated using the standard deviation in the historical outcomes for CPI inflation and GDP growth.

9 One alternative is the t distribution which has heavier tails, which may be more realistic if only a few large forecast errors have been observed.

is centred round the respective point forecast from the Riksbank. For example, a forecast interval is calculated with probability $(1-\alpha)$ in accordance with the following equation

$$y_t^{(h)} \pm z_{\alpha/2} \times \text{RMSE}(h), \quad h = 1,\ldots,H, \tag{1}$$

where $y_t^{(h)}$ is the Riksbank's point forecast of the variable y_{t+h} at time t, $z_{\alpha/2}$ is the $\alpha/2$ percentile in the normal distribution with expected value 0 and standard deviation 1. Accordingly, $z_{\alpha/2}$ is the value that in a standardised normal distribution has probability mass $\alpha/2$ to the left of point $z_{\alpha/2}$. Thus for a 90 per cent interval we have $\alpha = 0.1$ and $z_{\alpha/2} = z_{0.05} = 1.645$. The RMSE calculations are updated about once a year and for the Monetary Policy Report of July 2009 were based on forecast errors for the period 2000 to 2007.

FORECAST INTERVALS FOR REPO RATE FORECASTS

Since February 2007 the Riksbank has also published a repo rate forecast and 16 forecast paths have been presented since then. The repo rate forecast is also presented with a forecast interval, see Figure 1. The period in which the Riksbank has been forecasting the repo rate is, however, too short for making reliable RMSE calculations of the spread in the Riksbank's own forecast errors for the repo rate.[10] To achieve agreement with the calculation of the forecast interval for inflation and GDP it would of course be best to use the Riksbank's own repo rate forecasts, but for the time being the Riksbank must use an alternative method to calculate the spread of forecast errors for repo rate forecasts. By using the market implied forecast for future short-term interest rates (see next section) we can approximate the Riksbank's forecast capability, assuming that the historical forecast accuracy of market participants and the Riksbank are equivalent.[11]

The problem of having too few observations of forecast errors will disappear as more forecast errors can be recorded. In the near future the Riksbank can start to include its own forecast errors from the repo rate forecasts when calculating the RMSE, particularly for the shorter forecast

[10] The Riksbank's first repo rate forecast was made in February 2007 and runs to the first quarter of 2010. Since the quarterly mean for the repo rate in the first quarter of 2010 is not yet known it is not yet possible for example to compute any forecast error for the longest forecast horizon. Hence it will take a long time before the longer forecast horizons have sufficient outcomes to compare the forecasts with so as to be able to gain an idea of how great the forecast error spread is.

[11] In the short term, however, the Riksbank should have an advantage as regards the repo rate forecast, i.e. the forecast uncertainty should be less for the Riksbank than for the market participants. This indicates that the forecast intervals for the shortest forecast horizons should be based on the Riksbank's own forecast accuracy as soon as there is sufficient data for this.

horizons. For a long transition period forecast errors from both market pricing forecasts and the Riksbank's forecasts can be used together to measure the historical forecast accuracy for the repo rate.

The academic literature proposes an alternative, more forward-looking method to estimate uncertainty of short-term interest rates that is based on using interest rate options in various ways (see for example Svensson and Söderlind (1997) and Aguilar and Hördahl (1999)). Since the market price of options reflects the need of market participants to insure themselves against large fluctuations in interest-related securities it also gives a picture of the expected uncertainty during the validity period of the option contract. Based on the pricing of interest options it is therefore possible to calculate an implied probability distribution for a short-term interest rate at a given time. This probability distribution can then be used to calculate and illustrate forecast intervals. Unfortunately the Swedish market for this type of interest rate derivatives is relatively small, which means that there is an insufficient amount of reliable price data.

MARKET IMPLIED FORECASTS

According to the theory of effective financial markets and the expectations hypothesis, a market rate for a bond with a two-year maturity will for example reflect the expected yield from investing the money at the overnight rate day to day for two years. The yield curve, which shows how interest rates differ for different maturities at a given point in time, can therefore provide information on how market participants believe the repo rate will develop in future. By calculating implied forward rates the market participants' average forecast for short-term interest rates is obtained. According to the expectations hypothesis, the built-in expectations of the forward rates can be interpreted as the market participants' collective mean value forecast of the future interest rate level, given that the participants have rational expectations.[12] This means that market rates reflect the mathematical expected value of the market's overall forecast distribution, after excluding risk and maturity premiums. It is these expectations that are used as an approximation instead of the Riksbank's own repo rate forecasts, since this allows forecast errors to be studied a long way back in time.

[12] The overall mean value will thereby be a weighted volume of the individual participants' expected short-term interest rate outcome, on the basis of how much money they invest to make the interest rate higher or lower. If the largest investors in the fixed income market build on the belief that the interest rate will rise in the future then forward rates will also indicate higher interest rates. This means that the largest value participants' mean value forecasts will carry more weight than those of participants with smaller investment portfolios.

However, there is an important drawback associated with using forward rates to forecast the repo rate. Both academics and market participants have noted that market rates are not only based on expected repo rate development, which means that the expectations hypothesis does not give the whole answer to how a market rate is priced; see for example Campbell and Shiller (1991) and Alsterlind and Dillén (2005). What has been noted is that the interest rates, apart from the expected overnight rate, also include compensation to investors for any risks or costs associated with tying up their lending through longer maturities. This form of compensation is usually called risk and maturity premiums.

If the forward rates deviate to a certain extent from the expectations hypothesis, the interest rates should be adjusted for the different premiums causing the deviation. The premiums contain no information about market participants' expectations of the overnight rate and must therefore be excluded from forward rates to obtain the market participants' implied overnight rate forecast. Unfortunately, it is difficult to calculate the size of these premiums. We can observe a market rate for a certain maturity, for example a government bond rate, but there is no simple way to determine how much of the market rate reflects the expected average overnight rate and how much is some form of premium.

To identify forecast errors for forward rates the Riksbank at first used to opt for the simple assumption that the premiums for a specific forecast horizon can be estimated as the average forecast error for the unadjusted forward rates with corresponding maturities; see article in the Monetary Policy Report 2007:1. If, for example, the forward rates for four quarters ahead on average exceeded the realised repo rate outcome by 0.30 of a percentage point during the evaluation period, 0.30 of a percentage point was excluded from all forward rates with a maturity of four quarters. Adjustment of forward rates in this way was intended to make them better reflect the market participants' repo rate forecast. The deviations between the mean value adjusted forward rates and the repo rate path in the period 1998 to 2005 are the forecast errors that were used by the Riksbank to calculate forecast intervals for its repo rate forecasts up until September 2008.[13]

Assuming that the premiums in forward rates are equal to the entire average forecast error does not have any strong theoretical support. If the fixed income market participants have rational expectations one could expect that the average forecast error would be zero, which would certainly mean that an average deviating from zero can reflect the average

13 The method is outlined in the article "Calculation method for uncertainty bands" in the Monetary Policy Report 2007:1.

size of the premium. However, it is important to observe that the average forecast error can deviate from zero even if there were no premiums. One reason may be that the size of the sample is so small that it only covers one or two economic cycles. The average forecast error therefore includes both forward premiums and consistent forecast errors (positive or negative) during the period of evaluation.

In an attempt to improve the assumption about forward rate premiums and estimation of these premiums the Riksbank has used a model for the yield curve where risk premiums can be estimated (see Appendix for a more detailed description of the model). The model uses Swedish government bond rates in the form of an estimated yield curve for zero coupon rates. On the basis of theoretical relationships and interest rate movements from January 1998 to February 2008 the model identifies three underlying factors that have driven the changes in the yield curve. With the help of these three statistical factors the model can also identify average forward rate premiums for different maturities. The market participants' short-term interest rate forecasts implied in market rates of Swedish government bonds can thus be estimated by excluding forward rate premiums from forward rates. Since October 2008 it is the forecast accuracy of these adjusted forward rates that generates the RMSE values that determine the width of the forecast intervals for the repo rate forecast in Figure 1. The RMSE values are shown in Table 2.

TABLE 2. DIFFERENT FORECAST HORIZONS' RMSE FOR FORWARD RATE FORECASTS FOR ONE-MONTH RATE, JANUARY 2001 – FEBRUARY 2008.

	Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	Q10	Q11	Q12
Forward rates	0.17	0.33	0.50	0.71	0.93	1.12	1.28	1.42	1.54	1.60	1.66	1.74

ALTERNATIVE WAYS OF CALCULATING FORECAST INTERVALS

In the above we have given an account of the Riksbank's method of calculating forecast intervals on the basis of historical forecast errors. Here we will give a very brief account of two alternative ways of calculating uncertainty intervals: forecast intervals from formal probability models, and forecast intervals produced through expert assessments.

Given a formal probability model it is possible to calculate the full forecast distribution of a system of variables, possibly with the help of simulation methods. This distribution is simultaneous over both variables and forecast periods (see below). If Bayesian methods are used to estimate the model, parameter uncertainty in the model it is possible to take into account, or even some form of model uncertainty.

Scientific studies of the forecasting performance of economic and statistical models have often focused on the accuracy of the point forecast. Here is a short summary of the results of these studies:

- In the class of statistical models, simple models are usually equally good or better than more complicated models.
- Older structural economic models with microfundamentals (such as the Real Business Cycle (RBC) models, see King et al. 1988) perform worse than simple statistical models (Zimmermann, 2001).
- Modern structural economic models with micro fundations (such as the Dynamic Stochastic General Equilibrium (DSGE) models; see for example Christiano et al., 2005) are essentially just as good as statistical forecasting models (Smets and Wouters, 2003, Adolfson et al., 2007c, Edge et al., 2008), but structural models have are simpler to interpret and provide greater opportunities to calculate forecasts conditional on well-defined scenarios.
- Only at the very shortest forecast horizons do advanced expert assessments (for example the Federal Reserve's and Sveriges Riksbank's inflation forecasts) give more correct forecasts than statistical and structural models (see for example Sims, 2002 and Adolfson et al., 2007a).

Evaluations of forecast intervals and forecast distributions have started relatively recently to take a place in the macroeconomic literature, see for example Cogley et al. (2005), Adolfson et al. (2007c), Clark (2009), Jore et al. (2009) and Giordani and Villiani (2009). An important conclusion of these studies is that the disturbance variability has fluctuated substantially over time and that macroeconomic models therefore need time varying disturbance variances if the forecast interval is in fact to reach the probability coverage intended (for example 90 per cent for a 90 per cent interval).

The forecast intervals used by the Bank of England are of an entirely different nature. The Bank of England describes its method as the 'best collective judgement', which should be interpreted as a consensus of expert assessments. It would be going too far to discuss the model and expert-based forecast distributions in detail. We content ourselves here by very briefly mentioning the most important advantages and disadvantages of the two methods:[14]

[14] See also Adolfson (2007a) for a more detailed discussion and examples of episodes in the work of the Riksbank that the different approaches have treated in different ways.

- Advantages of model-based methods:
  - Provide a comprehensible intellectual framework and handle complex relationships in a system of endogenous variables.
  - Transparency. It is relatively easy after the event to understand the origin of the forecast and to learn from past mistakes.
- Disadvantages of model-based methods:
  - Formal models may be overly simple, thereby giving misleading results or forecasts.
  - Formal models find it difficult to take into account recently received information about the economy, particularly when this information is in a form that is difficult to adapt to the model structure.
  - Forecast intervals from formal models are often too narrow, since these intervals are based on a given model and therefore disregard the uncertainty of the model's specification (Chatfield, 1993).
- Advantages of expert assessments:
  - Can take into account recently received information in principle in any form (given that the expert can interpret and process the information)
- Disadvantages of expert assessments:
  - Not transparent. The expert seldom provides enough information about how he produces the forecast to enable an assessment of his mistakes. For the same reason it is difficult to learn from the expert's mistakes.
  - It is very difficult for an expert to handle systems of (endogenous) variables without an explicit model.
  - Experts often give forecast intervals that are too narrow; see for example Lawrence et al. (2006) which gives a number of reasons for this excess optimism.

Expert assessments are to a great extent the direct antithesis of model-based methods, and much could be gained if the two could be combined. It is already well-known how different model-based methods can be combined in the work of forecasting (for example via Bayesian model averaging, see Hoeting et al., 1999), and a certain amount has been written on how to combine expert assessments (French and Insura, 2000), but no practical, explicit and rigorous method of combining models and expert assessments has yet been proposed.

Improvements in the Riksbank method using historical forecast errors

TIME VARYING WIDTH OF FORECAST INTERVALS

One characteristic of the RMSE-based intervals is that their width are independent of the state of the economy. Hence uncertainty is assumed to be the same, regardless of whether we are in a downturn or an upturn, or even in times of economic crisis, which can be regarded as a disadvantage. However, it must be said that the most frequently used economic and statistical models are linear, with constant parameters and accordingly have exactly the same characteristic.[15]

In principle it is possible to generalise the method based on historical forecast errors so that the variance of forecast errors follows a model, for example by letting the variance of forecast errors (in logarithms) be time varying in accordance with an autoregressive process or by modelling the forecast error variance as a function of macro variables. The problem is that there are relatively few historical forecast errors available and estimation of these more complex models is therefore probably far too uncertain.

Nevertheless, it is natural to wonder whether RMSE-based intervals can be modified so that the interval width is dependent on the state of the economy. One obvious solution is to multiply the RMSE figures by an uncertainty factor which is, for example, greater than one in periods of extra uncertainty. This factor can be determined subjectively by the decision-maker on the basis of his/her perception of the current uncertainty in the economy. The Riksbank has previously used a method with a similar idea (Blix and Sellin, 1999).

A more objective and transparent method is to estimate a model with time-varying variance for the time series itself rather than for the forecast errors for this variable. This gives an idea of the uncertainty of the variable about which a forecast is being attempted, and this uncertainty can then be used for example to increase the RMSE figures in times of extra uncertainty. Assume for example that we are modelling GDP growth with an AR process in which the residuals follow a stochastic volatility model:

[15] Even in models with time-invariant parameters some time variation will of course arise in the parameter *estimates* over time.

$$y_t = \mu + \sum_{k=1}^{K} \phi_k (y_{t-k} - \mu) + \lambda_t \varepsilon_t,\ \varepsilon_t \overset{iid}{\sim} N(0,1), \qquad (2)$$

$$\ln\lambda_t^2 = \ln\lambda_{t-1}^2 + \nu_t,$$

where λ_t is the time varying standard deviation for the disturbances. The innovations to volatility, ν_t, are assumed to be independent $N(0,\psi)$. It is simple to generalise this model so that $\ln\lambda_t^2$ follows a general AR process, but we will focus here on the most common model where $\ln\lambda_t^2$ is a random walk. A Bayesian estimate $\hat{\lambda}_t$ can be calculated for λ_t for $t=1,\ldots,T$.[16] A measure of something that could be called the *relative volatility* at time T can now be defined as

$$\kappa_T = \frac{\hat{\lambda}_T}{n^{-1}\prod_{t=t_1}^{t_2} \hat{\lambda}_t^{1/n}},$$

where t_1 and t_2 are the opening and closing quarter for the time period in which the RMSE figures for one-stage forecasts are calculated, and $n = t_2 - t_1 + 1$ is the number of quarters during this period. Hence κ_T measures the volatility at time T in relation to the geometrical mean volatility during the period used to calculate the RMSE.

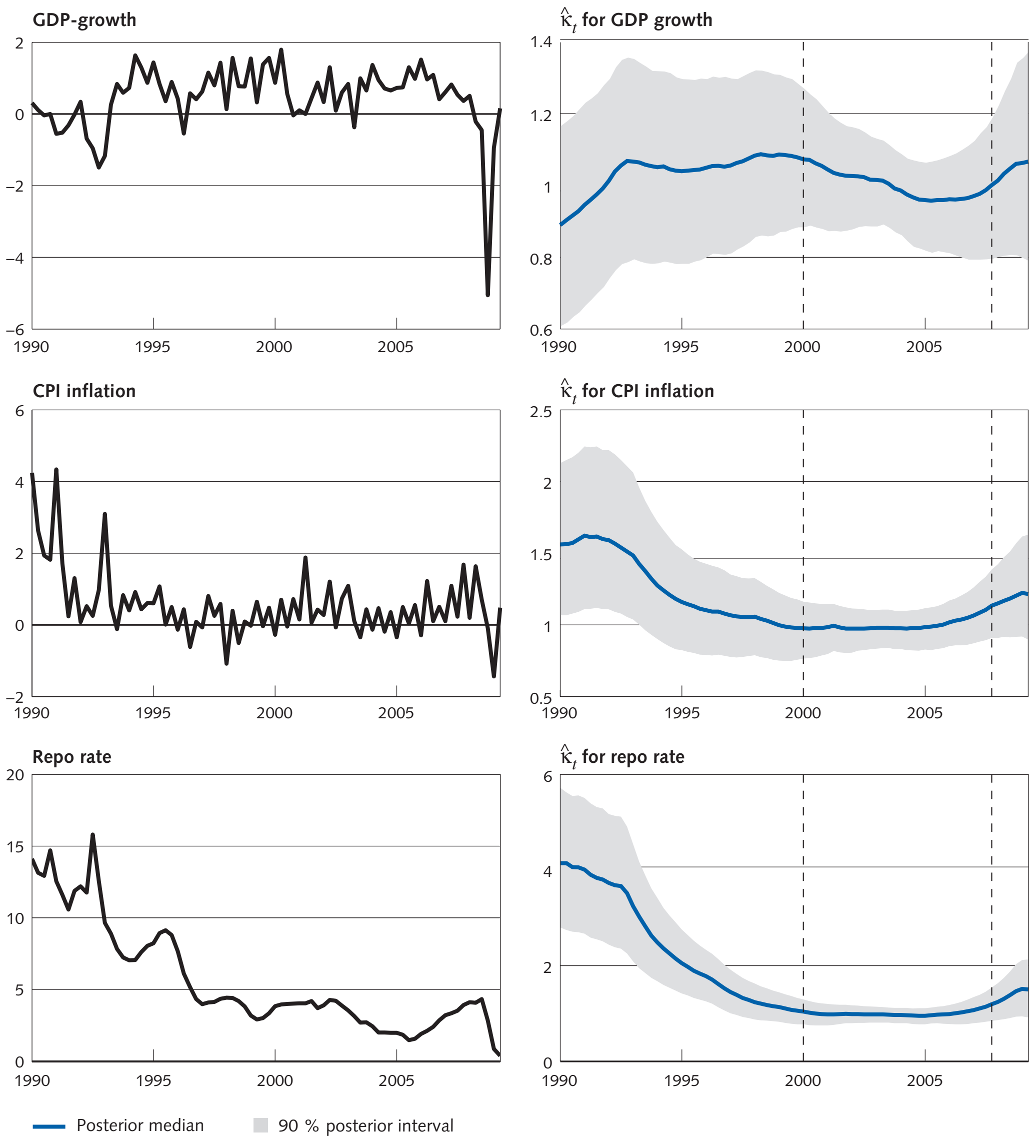
RMSE-based intervals can now be calculated in accordance with the method presented above, but replacing $\text{RMSE}(h)$ by $\kappa_T \cdot \text{RMSE}(h)$. Note that the RMSE values are multiplied by the same factor regardless of forecast horizon, which is analogous to multiplying the disturbance variance in a linear model by a constant.

Figure 2 shows the estimation results for κ_t for the quarterly percentage changes of seasonally adjusted GDP and CPI as well as for the repo rate, during the period between the second quarter of 1980 to the second quarter of 2009.[17] The results show that volatility has in principle been constant for GDP growth, but varied substantially for inflation and the repo rate. Figure 1 shows fairly clear indications that the long period of successively decreasing volatility of inflation and the repo rate was broken 1–2 years ago. The median in the posterior distribution for κ_{2009Q2}, the relative volatility for the second quarter of 2009, is 1.10, 1.27 and 1.55 for GDP growth, inflation and the repo rate. This means that the RMSE figures for example for inflation in Table 1 should increase by 27 per cent

[16] Using Bayesian estimation methods (see for example Clark, 2009) the posterior distribution of the entire sequence $\lambda_1,\ldots,\lambda_T$ can be calculated based on data up to and including period $t = T$. It would be going much too far to give all the details of the estimation here. The prior distribution of the time-invariant parameters is the same as in Villani (2009). The most important parameter is ψ, the innovation variance. We follow Giordani och Villani (2009) here and use an inverse gamma distribution as prior for ψ with expected value 0.01 and 10 degrees of freedom, which implies a reasonable time variation.

[17] Data up to and including the fourth quarter of 1988 are used as training observations to create an a prior distribution for λ_0 and the remaining observations are utilised to estimate the model.

Figure 2. Left column: Time series of quarterly growth of GDP and CPI, as well as the repo rate. Right column: Posterior median ($\hat{\kappa}_t$) and the 90 per cent interval for the relative volatility, κ_t. The start and end dates for the RMSE sample are marked with vertical broken lines.



due to the extra uncertainty today compared with the period 2000-2007. The increase in the RMSE figures for the repo rate of 55 per cent seems to be on the large side. One reason for the substantial fluctuations in volatility may be the assumption of normally distributed disturbances in the model in Equation 2, which results in extreme observations (called outliers) being overinterpreted as a drastic change in variance. Another problem with the model is that changes in variance are assumed to be frequent and consequently estimated as small. The variance in the repo rate seems rather to have been constant for long periods and then changed more abruptly on a few isolated occasions. The LASER model in Giordani and Villani (2009) is better at handling these problems, and may be an interesting alternative for this analysis. The LASER model allows non-normal disturbances and innovations, and the model's variances can be constant for long periods and then make bigger jumps.

The method above gives a scaling factor κ_T for each macroeconomic variable. An alternative is to scale the interval width of all variables with a common scale factor for the economy as a whole. A simple solution is to calculate a geometric mean value from the individual scale factors. A more advanced alternative is to estimate a vector autoregressive model with time varying covariance matrix that is time invariant up to common a scale factor:

$$x_t = \mu + \sum_{k=1}^{K} \phi_k (x_{t-k} - \mu) + \lambda_t \varepsilon_t, \varepsilon_t \overset{iid}{\sim} N(0, \Sigma)$$

$$\ln \lambda_t^2 = \ln \lambda_{t-1}^2 + v_t,$$

where x_t is a vector with observations on p time series at time t, ε_t is a p-dimensional vector with disturbances with covariance matrix Σ, and $v_t \overset{iid}{\sim} N(0, \psi)$ are the innovations to the univariate common volatility factor λ_t.

FORECAST INTERVALS THAT TAKE THE LOWER BOUND OF THE REPORATE INTO ACCOUNT

The exceptionally low interest rate levels that arose during the financial crisis create new problems. How does one design forecast intervals that take into account the fact that the repo rate cannot be negative? It must be pointed out that in principle it is possible to have a negative repo rate (Beechey and Elmér, 2009; Söderström and Westermark, 2009), but the repo rate can probably not lie too far below zero. Consequently, we will make the assumption that the repo rate floor is zero, but our method can easily be generalised for an arbitrary lower bound.

We have described above how the Riksbank's RMSE based forecast interval is based on the normal distribution, whose domain of possible outcomes is the interval $(-\infty, \infty)$. Thus the drawback is that the intervals for the repo rate can include negative values. This does not entail any problems in practice as long as the interest rate is not very low, since the probability of negative interest rates is then essentially zero. But at times of low interest rates the probability of a negative repo rate is substantial and the normal distribution assumption becomes more problematical. The Riksbank has tried two ways of tackling this problem. The first solution (used in the Monetary Policy Report in February 2009) consists of cutting the forecast bands for the repo rate in Figure 1 below zero. In the following Monetary Policy Report in July 2009 the forecast bands were retained in their original form, i.e. intervals containing negative values were allowed (see Figure 1). Intervals with negative values can be justified using two complementary arguments: i) the lower bound is not exactly zero, moderately negative interest rates cannot be ruled out (Beechey and Elmér, 2009) and ii) negative repo rates in the forecast distribution represent alternative monetary policy measures with the same effect *as though* the repo rate were negative, but are not to be interpreted as the repo rate actually being negative.

We will propose an alternative method here in the same spirit as the Riksbank's current method based on historical forecast errors, but which gives a forecast distribution in which the domain of possible outcomes is the interval $[0,\infty)$. There are many distribution families for non-negative random variables. We will focus here on *the gamma distribution*[18] whose probability density function is of the form

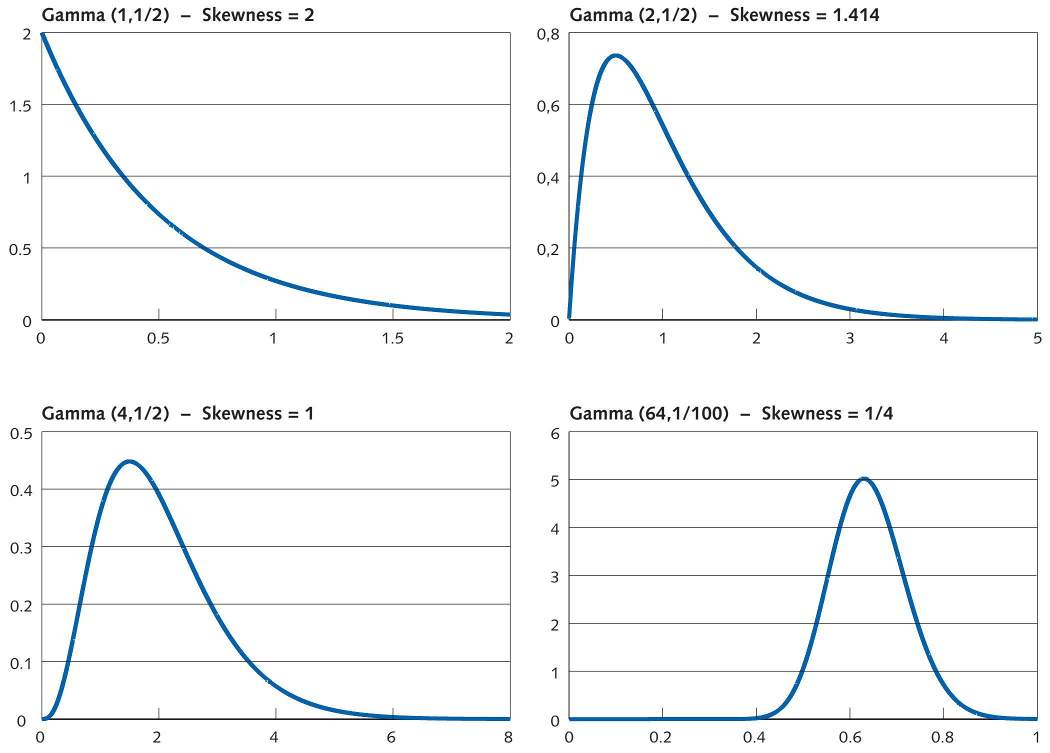
$$f(y;\alpha,\beta) = \frac{1}{\Gamma(\alpha)\beta^{\alpha}} y^{\alpha-1} \exp(-y/\beta),$$

where $\alpha > 0$ is called the *shape parameter* (because it determines the degree of skewness in the distribution) and $\beta > 0$ is the *scale parameter* (because if Y is $Gamma(\alpha,\beta)$ distributed then $cY \sim Gamma(\alpha,c\beta)$, where c is a scaling constant). Figure 3 shows some examples of distributions that are included in the gamma distribution family. The gamma distribution has an expected value $\alpha\beta$ and variance $\alpha\beta^2$. An important characteristic of the gamma distribution is that it converges towards the normal distribution when $\alpha \to \infty$. It can also be shown that the gamma distribution's skewness is $2/\sqrt{\alpha}$, and that the skewness is therefore small when the expected value is large in relation to the standard deviation, which for example is the case when the repo rate is close to its long-term equilibrium level. In times of normal interest rate levels the gamma distribution is therefore almost symmetrical. Note also that the gamma distribution can be easily generalised to the case of an arbitrary lower limit. If the lower limit is u then $y + u$ follows a generalised gamma distribution on the interval $[u,\infty)$.

---

18 One alternative is a lognormal distribution. This distribution is, however, very skewed, even when the distribution mass is a long way from zero. This means that the lognormal distribution will be considerably skewed even when the repo rate is in equilibrium, which may be seen as a disadvantage.

Figur 3. Some examples of gamma distributions



We will now describe how historical forecast errors can be used to estimate the gamma distribution parameters. The Riksbank's prediction interval for a given forecast horizon can be seen as an estimate of a normal distribution $N(\mu_h, \sigma_h^2)$ where

$$E_t(Y_{t+h}) = \mu_h = y_t^{(h)}$$

$$Std_t(Y_{t+h}) = \sigma_h = \text{RMSE}(h).$$

Expressed in words it can be said that the Riksbank matches the location of the normal distribution μ_h with the point forecast, while the standard deviation of the normal distribution σ_h is coupled with matched RMSE.

If we now assume that the forecast distribution h quarters ahead is given by a $Gamma(\alpha_h, \beta_h)$ distribution, then its parameters can be calculated analogously (see Appendix B) by matching i) the expected value in the gamma distribution with the Riksbank's forecast and ii) the standard deviation in the gamma distribution with the RMSE for historical forecast errors[19]. In the gamma case, however, it makes a difference whether the Riksbank's repo rate forecast is seen as an expected value or a median. In the standard theory of modern monetary policy analysis 'certainty equivalence' prevails, and therefore only the expected value forecast for the repo rate is of consequence for economic agents' decisions; other elements play no part (Woodford, 2003). This result appears to be the solution to our problem: the Riksbank's forecast is an intention and according to the result on certainty equivalence the Riksbank should therefore communicate an expected value forecast. Unfortunately certainty equivalence does not apply when taking into consideration the lower bound of the interest rate, since this restriction makes the model non-linear. Precisely where the choice of point forecast makes a difference (that is when the repo rate is close to zero and the distribution probably is skewed) we cannot rely on the support of economic theory in the area.[20] The Riksbank does indeed communicate that its point forecast is an expected value, but the Riksbank's informal process for producing the forecast means that other types of point estimates, such as a median forecast, cannot be ruled out; see for example the monetary policy minutes of the Executive Board meeting of 1 July 2009. Box 1 discusses the terms under which various point forecasts are optimal from a statistical perspective.[21] In Appendix B we provide detailed solutions for the gamma model in the two cases where the Riksbank's forecast is an expected value or a median respectively.

Box 1 – How to choose an optimal point and interval forecast?

OPTIMAL POINT FORECAST

The choice between different point forecasts, such as expected value, median or type value, is determined by one's loss function, i.e. the loss one makes when forecasting a variable with the value $\hat{y}$ and the actual outcome is y. The classical example is the squared loss function

$$L(y, \hat{y}) = (y-\hat{y})^2$$

[19] It can be questioned whether the spread of historical forecast errors is really relevant in this new situation with low interest rate levels, without precedent in historical data. The question then arises whether uncertainty about the repo rate is currently greater or smaller than normal. The lower bound of the rate indicates that uncertainty is less, since the rate in principle cannot be much lower, but on the other hand the repo rate may need to be raised faster than expected if the financial crisis and recession are more short-lived than expected.

[20] Certainty equivalence can, however, serve as a good approximation and this is worth studying more carefully.

[21] It is important to point out that using modern simulation methods we can in principle always calculate the entire forecast distribution and it is not self-evident that we must in fact opt to reduce this distribution to one or more summary measurements, such as a point forecast. An important reason for a central bank deciding to stress a point forecast rather than an entire forecast distribution is that it simplifies communication to the market and the general public. But when the forecast distribution is asymmetrical it is difficult to get away from the fact that a point forecast gives a very rough and perhaps even misleading summary of the distribution.

that results in the forecast distribution's expected value $E(y)$ being the optimal point forecast. If the loss function is instead linear in the absolute forecast error

$$L(y, \hat{y}) = y - \hat{y}$$

then the median is the optimal point forecast. One may think that the mode, the most probable value, is a natural point forecast, but in that case it should be remembered that this forecast is only optimal with the rather peculiar all or nothing loss

$$L(y, \hat{y}) = 0 \text{ if } y = \hat{y}, \text{ but } L(y, \hat{y}) = 1 \text{ if } y \neq \hat{y},$$

i.e. that one suffers the same loss regardless of the size of the forecast error, except from when a completely accurate forecast is given and the loss is zero. Another interesting loss function is the lin-lin loss, where underestimation and overestimation are treated asymmetrically:

$$L(y, \hat{y}) = c_1 \text{ if } \hat{y} \leq y, \text{ but } L(y, \hat{y}) = c_2 \text{ if } \hat{y} > y,$$

where c_1 and c_2 are constants. In that case the optimal forecast $c_1/(c_1 + c_2)$ is the percentile in the forecast distribution. If, for example, the loss of underestimating inflation is twice that of overestimating it ($c_1 = 2c_2$) then the 66th (2/3) percentile in the forecast distribution is the optimal point forecast.

OPTIMAL FORECAST INTERVAL

A slightly fuller summary of a forecast distribution is given by a probability interval. For intervals too there is more than one type to choose from for a given interval probability. Perhaps the most common interval is one that excludes as much probability mass below the lower limit as above the upper limit. This *centred* interval is optimal if the loss is of the form (Wallis, 1989)

$$L(y,[a,b]) = \begin{cases} c(b-a)+d(a-y) & \text{if } \ y<a \\ c(b-a) & \text{if } \ a \leq y \leq b \\ c(b-a)+d(y-b) & \text{if } \ y>b \end{cases}$$

that is to say if the loss is linear in the distance between outcome y and interval .

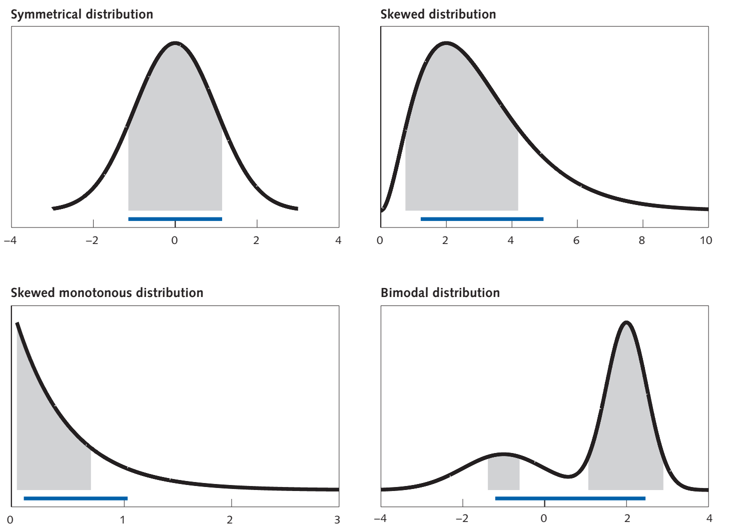
If instead the loss is in the all or nothing form

$$L(y,[a,b]) = \begin{cases} c(b-a)+d & \text{if } \ y<a \\ c(b-a) & \text{if } \ a \leq y \leq b \\ c(b-a)+d & \text{if } \ y>b \end{cases}$$

then the interval $[a, b]$ is optimal if the endpoints a and b have the same density in the forecast distribution (Wallis, 1989). This interval has the shortest length for a given coverage probability and includes the points with highest density, and is therefore called the *Highest Posterior Density* (*HPD*) interval.

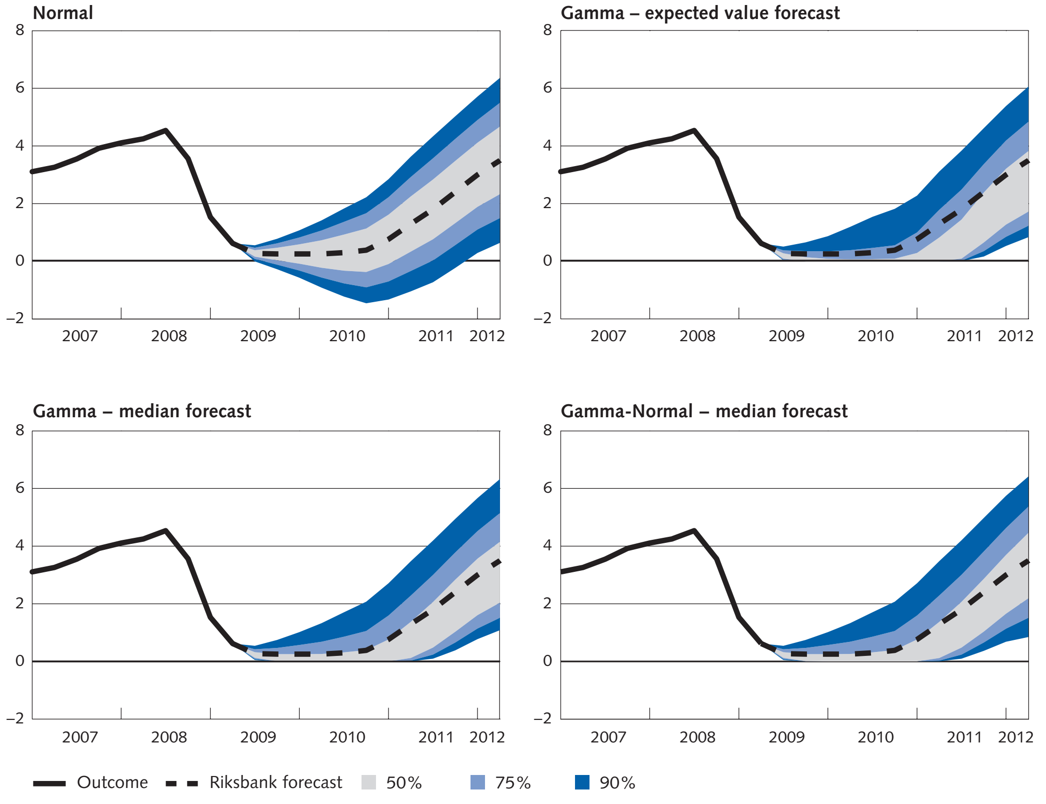
Figure B1 illustrates these two interval types for various forecast distributions. If the forecast distribution is symmetrical, centred intervals coincide with HPD intervals. This is probably the reason that the choice of forecast intervals is rarely discussed explicitly. But for asymmetrical distributions the interval type plays an important role. Note also that the centred interval can exclude points with very high density (the distribution on the bottom left of Figure B1), and include points with very low density (the distribution on the bottom right of Figure B1). The example of the skewed monotonous distribution in Figure B1 is particularly relevant here since this distribution appearance arises when the gamma distribution is used to model the forecast distribution for the repo rate at very low interest rate levels. Figure B1 shows that 75 per cent HPD intervals in this example will include the zero repo rate case, but that this point will fall outside the centred 75 per cent interval.

Figure B1. Comparison of 75 per cent HPD intervals (shadowed areas) and 75 per cent centred intervals (blue horizontal line)



One example of a forecast interval from the gamma model is shown in Figure 4. The top left graph of the Figure replicates the Riksbank's point forecast and forecast interval for the repo rate from the normal model published in the Monetary Policy Report of July 2009. The corresponding forecast interval for the gamma model is shown on the right of this graph, assuming that the Riksbank's forecast is an expected value. Notice that the forecast interval does not include negative interest rates (a consequence of the gamma distribution), and that the lower limit of the interval can be zero (a consequence of the highest posterior density (HPD) interval used in the figure, see Box 1. The actual forecast distribution has the shape illustrated in the lower left corner in Figure B1 when the repo rate forecast is close to zero). The pronounced skewness of the distribution (cf. for example the 75 and 90 per cent forecast intervals) is a consequence of the very low expected value (the Riksbank's point forecast) in combination with a relatively large standard deviation (given by historic RMSE).

Figure 4. Reconstruction of the forecast distribution for the repo rate in the Monetary Policy Report of July 2009 under various assumptions on distribution form and various assumptions on the Riksbank's point forecast



The graph in the lower left corner is also based on the gamma distribution, but now under the assumption that the Riksbank's repo rate forecast is a median forecast. Under this assumption the distribution is considerably less skewed, and has a generally reasonable appearance. A slightly less attractive side-effect of the gamma distribution is that it is still somewhat skewed even when the repo rate reaches more 'normal' levels.

The final subgraph in Figure 4 shows a way of retaining the attractive characteristics of the gamma distribution at low repo rate levels while more rapidly approaching a symmetrical distribution when the repo rate assumes more normal levels. This forecast distribution is a hybrid of a gamma distribution and a normal distribution:

$$p(y_{T+h}|y_1,y_2,\ldots,y_T) = \pi_h \cdot I_0(y_{T+h}) + (I-\pi_h) \cdot Gamma(y_{T+h}|\alpha_h,\beta_h),$$

where $N(y|\mu,\sigma^2)$ designates the probability density function for a $N(\mu,\sigma^2)$ distribution and $Gamma(y|\alpha,\beta)$ is the density function for a gamma distribution. The weight of the normal distribution $\omega(\hat{y}_{T+h})$ is a logistics function of the Riksbank's point forecast

$$\omega(\hat{y}_{T+h}) = \frac{\exp(c_0 + c_1\hat{y}_{T+h})}{1+\exp(c_0 + c_1\hat{y}_{T+h})},$$

where $c_0 = -10$ and $c_1 = 3$, which gives the function in Figure 5. When the repo rate forecast is close to zero the forecast distribution is in principle the same gamma distribution as before. The weight of the normal distribution then increases with the level of the repo rate and already at a rate of about 4.5 per cent the forecast distribution is in principle the same as the normal distribution.

Figure 5. Weight of the normal distribution component in the gamma-normal hybrid

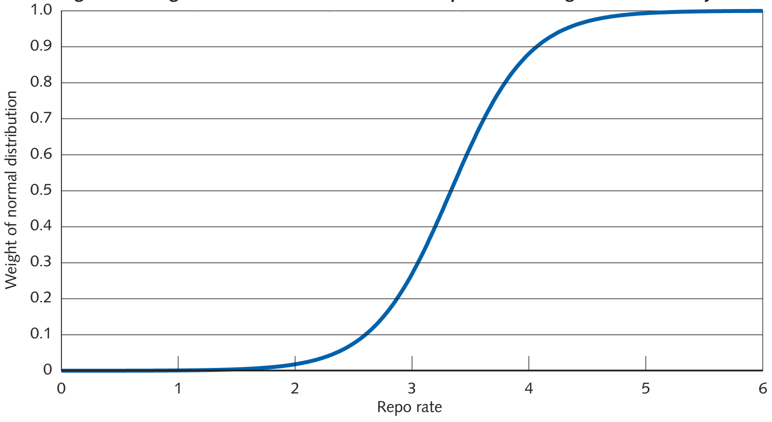


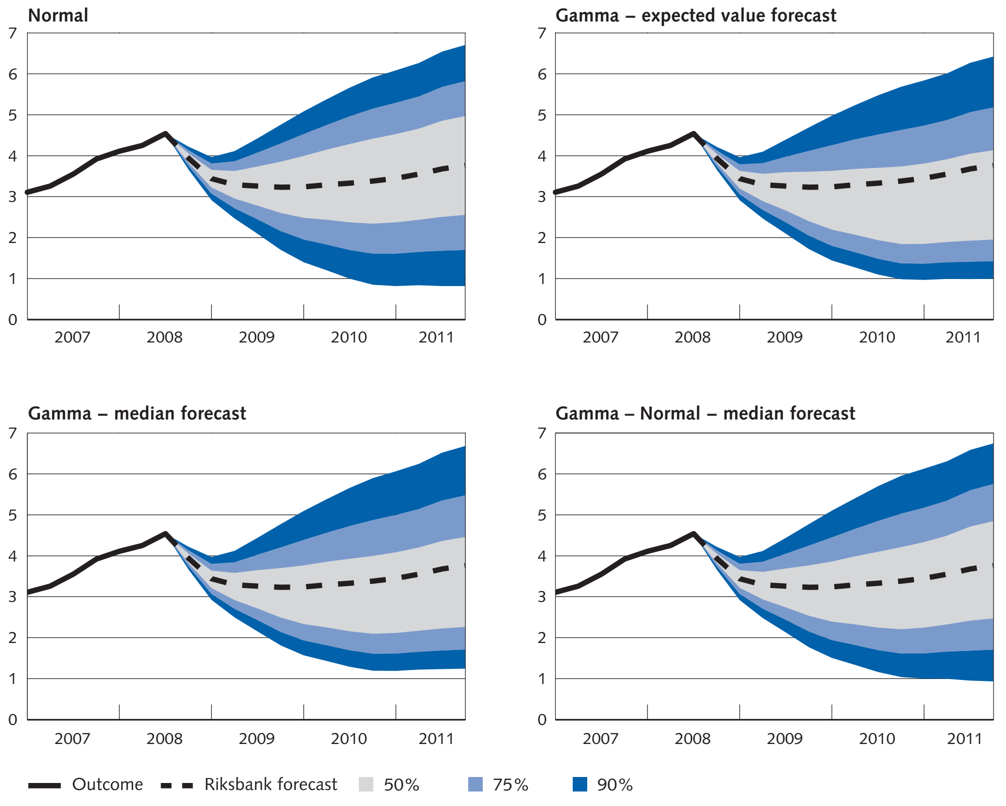
Figure 6 is a repetition of Figure 4, but now with the more normal interest rate levels that applied when the Monetary Policy Report 2008:3 was published. All four forecast distribution variants are relatively similar, but the two gamma distributions present some skewness that is not found in the normal or gamma-normal models.

An objection to the above analysis with the gamma distribution is that the outcome with a repo rate at exactly zero is not treated differently from any other point. It is possible to argue for the appropriateness of the more general distribution

$$p(y_{T+h}|y_1,y_2,\ldots,y_T) = \pi_h \cdot I_0(y_{T+h}) + (I-\pi_h) \cdot Gamma(y_{T+h}|\alpha_h,\beta_h),$$

where $I_0(y_{T+h})$ is a point mass at zero with probability π_h. This means that with probability π_h the repo rate is exactly zero and with probability $1-\pi_h$

Figure 6. Reconstruction of the forecast distribution for the repo rate in the Monetary Policy Report 2008:3 under various assumptions on distribution form and various assumptions on the Riksbank's point forecast

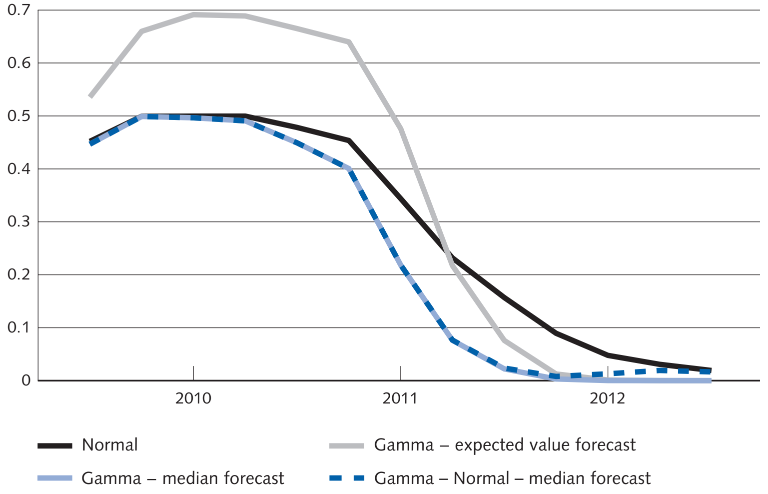


it follows the gamma distribution. The problem with this formulation is that it is very difficult to estimate π_h from historical predictions since the repo rate has never actually been zero historically. An obvious solution is to determine π_h subjectively and then estimate the parameters in the gamma distribution conditional on π_h in accordance with our earlier method.

Even if the gamma distribution does not provide a separate discrete probability to the outcome of a repo rate at exactly zero, it should be noted that this distribution still assigns a large probability to outcomes so close to zero that they are practically equivalent to a zero rate. Figure 7 displays the probability of a repo rate lower than 25 basis points for the four models in Figure 4. All four models therefore imply a substantial probability of a repo rate that is practically zero until the end of year 2010.

During the second half of 2009 the Riksbank has noted that the market repo rate forecast (calculated from implied forward rates, see above)

Figure 7.The figure displays the probability of a repo rate lower than 25 basis points f the four models in Figure 4

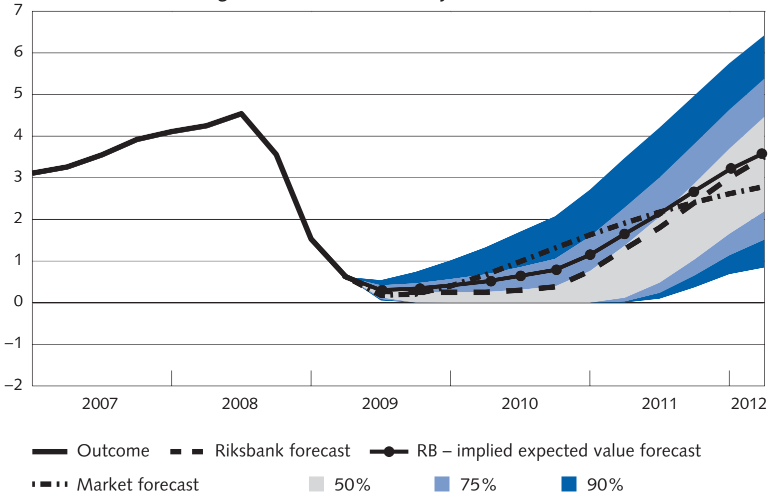


is higher than the Riksbank's point forecast. This could be seen as an indication that the market does not share the Riksbank's view of the macroeconomic outlook or that the Riksbank has not succeeded in gaining support for its intentions. However, it is fully possible that the Riksbank's and the market *point forecasts* can diverge despite having the same underlying forecast distribution, see Box 1. If, for example, the Riksbank's forecast is a median, while the market forecast is an expected value, the skewness of the gamma distribution when interest rates are low will lead to the expected value forecast being higher than the median forecast. Figure 8 compares the market repo rate forecast on two occasions with the Riksbank's implied expected value forecast for the gamma-normal model (where the Riksbank is assumed to publish a median forecast). It is clear that the market forecasts are considerably higher than the Riksbank's published forecast in the second year of the forecast period, but that the Riksbank's implied expected value forecast lies much closer to the market forecast in the same period of time. Hence a reinterpretation of the Riksbank's point forecast as a median rather than an expected value could explain a great deal of the gap between the market expectations and the Riksbank's forecast.

SIMULTANEOUS FORECAST BANDS

The forecast bands in Figure 1 are a number of forecast intervals, one for each forecast horizon, that are connected by lines. The forecast intervals for each forecast horizon are *marginal intervals*, over both variables and forecast horizons. This means that the intervals do not contain any

Figure 8. Comparison of the Riksbank and the market repo rate forecasts with expected value forecasts in the gamma-normal model. July 2009



information on relationships between variables (for example the correlation between GDP growth and inflation) or relationships over different forecast horizons (for example the correlation between inflation one or two years ahead respectively). In this section we will describe these facts in detail, as well as give an account of different types of forecast bands proposed in scientific literature to describe relationships over forecast horizons.

In situations with more than one variable, simultaneous probability distributions are used to describe the co-movement of the variables. From a simultaneous probability distribution of $p(y,\pi)$ for GDP growth (y) and CPI inflation (π) one can for example calculate the probability of negative growth $(y<0)$ *at the same time* as inflation exceeds the Riksbank's tolerance interval $(\pi>3)$. There are two important distributions that can be derived from a simultaneous distribution: conditional distributions and marginal distributions.

Conditional distribution is the distribution for inflation π given a certain value of GDP growth y and is denoted $p(\pi|y)$. This distribution is more geared to scenario analysis and can answer questions of the type: what is the forecast distribution of inflation *given* that GDP growth is zero per cent?

The *marginal distribution* for π is the distribution for inflation alone, taking in account all possible outcomes for GDP growth by means of probability weighting for these different outcomes.[22] It is important to

[22] The marginal distribution of inflation is calculated as $p(\pi) = \int p(\pi|y)p(y)dy$, where $p(y)$ is the marginal distribution of y.

understand that the marginal distributions can be derived from the simultaneous distribution, but not vice versa: it is not possible to recreate the simultaneous distribution from the marginal distributions. In other words, the marginal distributions say nothing of the dependence between variables; see the illustration in Figure 9 that shows that two bivariate distributions with different correlation coefficients can have identical marginal distributions. The uncertainty bands that the Riksbank presents in the Monetary Policy Report are marginal distributions for GDP growth, CPI inflation and the repo rate, hence they contain no information about the Riksbank's view of the future *covariance* between variables.

However, the marginal distributions must be consistent between variables and in this sense there is some relation between the variables remaining in the marginal distributions. If, for example, the repo rate is determined by a simple Taylor rule without regard to the real economy, i.e. $r_t = 1.5\pi$ then $\text{Var}(r_t) = 1.5^2\text{Var}(\pi_t)$ is applicable, in other words the interval width for the repo rate should be 1.5 times greater than for inflation.

In the same way as it is possible to speak of relationships *between variables* for a given forecast horizon, it is possible to speak of the relation of an individual variable *over the forecast horizons* (for example what is the probability of inflation exceeding 3 per cent in **both** a 1 and 2 year perspective?). But the uncertainty regions in Figure 1 are a number of *marginal* forecast intervals that are linked together with lines, which thus do not contain any information about covariance over forecast horizons.[23] Consequently the 90 per cent forecast bands in the Monetary Policy Report do *not* describe the area where the future outcome *path* will be with a probability of 90 per cent, since this event includes all the 12 forecast horizons simultaneously. The forecast bands in Figure 1 must be read forecast horizon for forecast horizon, and it may therefore be slightly misleading to link these marginal intervals by lines as in Figure 1, but this representation has been adopted by all central banks that present forecast intervals. From now on we will call the linked marginal intervals in Figure 1 *marginal bands*, to differentiate them from *simultaneous bands* that represent the simultaneous distribution over all forecast horizons.

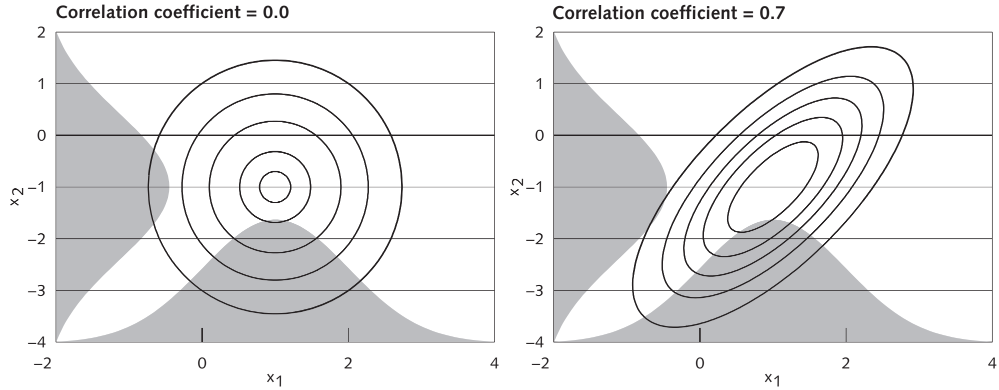
The actual simultaneous probability that an outcome path for example for the repo rate will fall inside the 90 per cent marginal bands in Figure 1 is considerably lower than 90 per cent. If, for example, we

23 The interval around for example the four-step forecast ($h = 4$) has thus been computed from the marginal distribution

$$p_T(y_{T+4}) = \iiint p_T(y_{T+1}, y_{T+2}, y_{T+3}, y_{T+4})\,dy_{T+1}\,dy_{T+2}\,dy_{T+3},$$

where the uncertainty around the outcomes y_{T+1}, y_{T+2} and y_{T+3} has been integrated out.

Figure 9. Illustration showing that two simultaneous distributions with different correlation coefficients can have identical marginal distributions. The simultaneous distributions are represented by elliptical contours of the same density and the marginal distributions are indicated by shadowed areas in the diagrams



assume an extreme case where a variable follows an independent process entirely without persistence then the probability that the outcome path will fall within the marginal bands is $0.9^{12} \approx 0.282$. To calculate the equivalent probability for a more persistent process, we simulate 1000 time series with 200 observations each from an autoregressive process of the first order (AR(1)):

$$y_t = \mu + p(y_{t-1} - \mu) + \varepsilon_t,\ \varepsilon_t \overset{iid}{\sim} N(0,\sigma^2), \qquad (3)$$

with $\mu = 2$, $\sigma = 0.25$ and $p \in \{0.25, 0.5, 0.75, 0.9\}$. For each simulated time series we estimate the AR(1) model in Equation 3 recursively over the entire random sample, and as of time $t = 100$ a point forecast is calculated in each time period with associated forecast interval 1–12 quarters ahead. The forecast interval is only calculated using the historical forecast errors that are available at the time of the calculation. The first forecast is made in period $t = 100$ and then uses all available forecast errors as of $t = 51$ up to and including period $t = 100$. This design is intended to imitate the Riksbank's way of calculating forecast intervals, with the important difference that here we know the data-generating process (but not its parameters). For each simulated time series we register the proportion of intervals that include the entire outcome path $y_{t+1},\ldots,y_{t+12}$, which is a simulation approximation of the forecast bands' simultaneous coverage probability. This exercise is similar to that in Table 3 in Jorda and Marcellino (2010), but here we take into account the estimation uncertainty in both the point forecast and the estimated RMSE figures in an attempt to better imitate the actual situation of the Riksbank.

The results of the simulations are reported in Table 3, which shows that the actual probability that the *entire* outcome path will fall inside all the 12 marginal intervals is very much lower than the coverage probability of the marginal intervals even for very persistent processes (see the lines called 'Marginal bands'; the results of the other lines in Table 3 are explained below).

One could consider supplementing Figure 1 with a corresponding graph with the simultaneous forecast bands. The problem here is to reduce a 12-dimensional simultaneous distribution to something that is drawn in a two-dimensional figure in the form of forecast bands. A simple approach is to utilise the Bonferroni inequality to create simultaneous forecast bands. Bonferroni bands with simultaneous probability $1-\alpha$ are calculated using a formula similar to that for the marginal interval (cf. Equation 1) [24]

$$y_t^{(h)} \pm z_{a/2H} \times \text{RMSE}(h), \tag{3.3}$$

but note that here we use the $a/2H$ percentile in the $N(0,1)$ distribution, where H is the maximum number of forecast horizons in the figure, i.e. $H = 12$ in Figure 1.[25] Where we in the previous case of 90 per cent marginal intervals used the value $z_{0.05} = -1.645$, here use the value $z_{0.05/12} = -2.638$ for a 90 per cent Bonferroni band. Bonferroni bands are conservative: a 90 per cent Bonferroni band has a simultaneous probability of *at least* 90 per cent (if the model is correct and its parameters known). Table 3 shows that the Bonferroni bands come very close to the target probability for the 75 per cent and 90 per cent bands, but give far too wide 50 per cent bands, particularly if the process is persistent. Higher order Bonferroni bands have been studied in Ravishanker et al. (1991) who found them more correct than the ordinary (first order) Bonferroni bands.

The Bonferroni bands are designed to control the simultaneous probability for outcome paths. Somewhat vaguely it can be said that a secondary effect of this is that the highest priority of these forecast bands is to prevent the outcome paths being outside the bands too often for one single forecast horizon; see Jorda and Marcellino (2010) for a more precise formulation. Jorda and Marcellino (2010) argue that this fixation on single forecast horizons may be suitable for certain applications in financial economics, but that it is less reasonable for macroeconomic analysis. They instead advocate Scheffé's *S*-method (Scheffé, 1959) for creating simultaneous forecast bands. Scheffé bands are designed to control the Maha-

24 Bonferroni's inequality says that $\Pr(\cap_{h=1}^{H} E_h) \geq 1 - \sum_{h=1}^{H} \Pr(\bar{E}_h)$, where E_h is the event in which the outcome y_{T+h} lies within the marginal interval at the forecast horizon h, $\bar{E}_h$ is the complementary event of E_h, i.e. that the outcome will lie outside the marginal interval. If the probability for each marginal interval is set at $1-\alpha/H$ we thus get $\Pr(\bar{E}_h) = \alpha/H$, and the simultaneous probability for the forecast band then fulfils the inequality $\Pr(\cap_{h=1}^{H} E_h) \geq 1-\alpha$, i.e. Bonferroni bands give a simultaneous probability of at least $1-\alpha$.

25 It should be mentioned that Bonferrroni bands **do not** assume independence over forecast horizons, which is too often wrongly asserted in the literature.

TABLE 3. SIMULTANEOUS PROBABILITY THAT ALL THE COMING 12 OUTCOMES WILL FALL WITHIN DIFFERENT TYPES OF FORECAST BAND.

50-per cent forecast band				
	$\rho = 0.25$	$\rho = 0.50$	$\rho = 0.75$	$\rho = 0.90$
Marginal bands	0.0006	0.0009	0.0046	0.0168
Bonferroni	0.5880	0.6142	0.6508	0.6909
Scheffé	0.0693	0.1916	0.3826	0.4896
Scheffé top-down	0.0083	0.0473	0.1868	0.2904
75-per cent forecast band				
	$\rho = 0.25$	$\rho = 0.50$	$\rho = 0.75$	$\rho = 0.90$
Marginal bands	0.0435	0.0609	0.1198	0.1857
Bonferroni	0.7622	0.7628	0.7879	0.7859
Scheffé	0.1537	0.3190	0.5287	0.5968
Scheffé top-down	0.1846	0.3635	0.5686	0.6295
90-per cent forecast band				
	$\rho = 0.25$	$\rho = 0.50$	$\rho = 0.75$	$\rho = 0.90$
Marginal bands	0.2912	0.3427	0.4153	0.4967
Bonferroni	0.8865	0.8830	0.8804	0.8825
Scheffé	0.2453	0.4530	0.6252	0.6956
Scheffé top-down	0.5759	0.7414	0.8252	0.8564

lanobis distance between the forecast path and outcome path, which means that this method lays more weight on deviations at more than one forecast horizon. In Table 3 we see that the Scheffé bands are too narrow at the 50, 75 and 90 per cent level, but that they become more correct at higher persistence levels. Note, however, that the Scheffé bands are not designed to control the simultaneous probability for outcome paths in the sense that we measure in Table 3. However, the Scheffé bands are much better than the Bonferroni bands at controlling the Mahalanobis distance between the outcome path and the point forecast; see the simulation results in Table 3 of Jorda and Marcellino (2010).

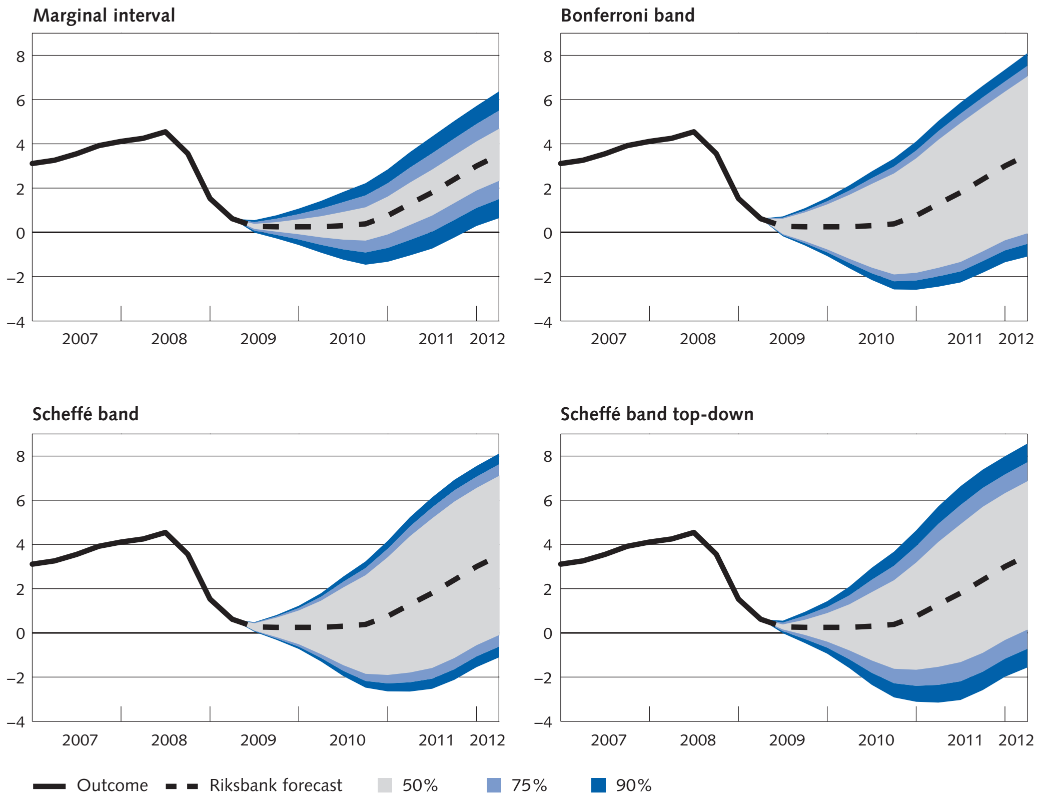
In the academic literature on forecast bands it is implicitly assumed that simultaneous forecast bands should always be presented. However, there are two good reasons for using marginal bands in practice: i) simultaneous bands have the disadvantage that it is not possible to identify for example the forecast interval for inflation 1 year ahead, ii) the width of simultaneous bands depends on the choice of maximum forecast horizon, H; in other words the simultaneous bands will be different if the Riksbank decides to present them for 1–8 quarters compared with 1–12 quarters. In the same way the simultaneous bands for inflation will be very different if they are presented as monthly outcomes ($H=36$) or quarterly outcomes ($H=12$). The latter problem is discussed in Jorda and Marcellino (2010) and they propose a top-down approach in which the simultaneous intervals no longer depend on H. The simulation results in Table 3 show that these modified Scheffé bands actually give a more correct simultaneous probability for the 75 per cent and 90 per cent forecast bands than the original Scheffé bands, but the top-down approach works less well for the 50 per cent forecast band.

Figure 10 illustrates the different forecast bands for the repo rate in the Monetary Policy Report of July 2009. All types of simultaneous bands are considerably wider than the marginal bands, particularly for the lower coverage probabilities 50 and 75 per cent.

Conclusions

We have described how Sveriges Riksbank computes forecast intervals for the repo rate, CPI inflation and GDP growth using a method based on the variation in historical forecast errors. The method is simple and easy to understand and has the advantage of not being dependent on a specific model. This means that the intervals also include uncertainty about the functioning of the economy. However, Sveriges Riksbank started to publish its own repo rate forecast only relatively recently, which means

Figure 10. Illustration of different ways of designing simultaneous forecast bands for the repo rate in the Monetary Policy Report, July 2009



that only a few forecast errors are available for this variable, particularly for the longer forecast horizons. We describe how the Riksbank has chosen to solve this problem by instead using the implied repo rate forecast, adjusted for forward premiums, from the financial market.

We have proposed a number of potential improvements to the Riksbank's current interval structure, including a method of introducing time variation in the width of the forecast intervals. This change has the advantage of allowing the forecast bands to be wider in times of more uncertainty, for example. The fact that the repo rate is very close to its lower bound (which is zero or slightly lower) also makes new demands on the forecast bands. We have therefore introduced a simple way of designing forecast intervals for the repo rate in the same spirit as the current method, but where the repo rate cannot be negative. Finally, we also discussed the pros and cons of different methods of designing forecast bands that describe the simultaneous uncertainty over all 12 forecast horizons.

We want to conclude this article by again pointing out that the Riksbank's forecast interval describes general economic uncertainty, hence differing from the point forecast, which should be understood as the Riksbank's intention; see Section 1. However, it must not be ruled out that the forecast bands will in future be supplemented or replaced by forecast bands with an intention interpretation. The Riksbank's main model for structural analysis, Ramses (Adolfson et al., 2007b), was implemented relatively recently with optimal monetary policy (Adolfson et al., 2008b), and could therefore be used for this purpose. It is an open question whether this model generates forecast bands with reasonable width or if the model needs to allow time variation in the disturbance variances to achieve correct interval probability.

Appendix A. Financial model of the yield curve for Swedish interest bearing government securities

The Riksbank uses a three-factor model to model how the yield curve for Swedish interest-bearing government securities develops over time; see for example Backus et al. (2000) for an examination of multi-factor models. According to the model, both the short-term interest rate and more long-term bond rates depend on three factors $x_t = (x_{1t}, x_{2t}, x_{3t})$ which are assumed to follow the model

$$x_{t+1} = \Phi x_t + \varepsilon_t,$$

where ε_t is an exogenous stochastic shock with diagonal covariance matrix Σ, and the shocks are here assumed to be independent innovations. The short-term interest rate is determined according to the equation

$$r_t = d_0 + x_{1t} + x_{2t} + x_{3t}.$$

In the model it is also assumed that it is not possible to make risk-free arbitrage profits between bonds with different maturities, and that the interest on all bonds follows the fundamental pricing relation

$$P_t = E_t(P_{t+1}M_{t,t+1})$$

which states that the bond price is the expected discounted future price of the bond. The discount is determined by the stochastic discount factor M, which is determined by the short-term interest rate, the price of risk λ, the shocks to x_t and the covariance matrix of the shocks

$$M_{t,t+1} = e^{-r_t - \lambda'\Sigma\lambda/2 - \lambda'\varepsilon_{t+1}}.$$

On the basis of these assumptions it can be shown that the interest y on zero coupon bonds with maturity n is also dependent on the three factors, the short-term interest rate, the risk price and the variance of the factors (see Ang and Piazzesi (2003) for a derivation)

$$y_t(n) = -\frac{1}{n}\left[A(n) + B(n)'x_t\right]$$

where

$$A(n) = A(n-1) + B(n-1)'\Sigma B(n-1)/2 - d_0 - B(n-1)'\Sigma\lambda$$

and

$$B(n) = B(n-1)'\Phi - 1.$$

Maturity premiums that depend on the risk price λ are included in $y_t(n)$. But because the model can identify risk premiums for different maturities the model can also be used to compute forward rates where the risk premium component is excluded. This forward rate f_t^* for short-term interest rate r in n time units can be written as

$$f_t^*(n) = d_0 + \Phi^n x_t.$$

By calculating $f_t^*(n)$ for all n up to the forecast horizon (36 months) and for all observed (monthly) forward rate curves since 1998 the model's interpretation of the market priced short-term interest rate forecast is obtained. $f_t^*(n)$ is the forecast used to calculate the RMSE values that describe the historical forecast accuracy of the repo rate.

The three underlying factors of x_t are not directly observable. To provide a statistical inference for the x_t-process the three factors are linked to observable measurement variables. The measurement variables used by the Riksbank are computed zero coupon rates (to avoid the complexity of coupon interest rates) for government bonds and survey responses concerning market analysts' repo rate expectations (Kim and Orphanides, 2005). The data is monthly and the short-term interest rate in the model is therefore the one-month rate, which is considered to be a decent approximation of the Riksbank's policy rate. The Kalman filter is used to infer the underlying factors of x_t on the basis of the observed measurement variables (Hamilton, 1994). Simultaneously with this filtering the model's parameters are estimated with the maximum likelihood method.

As the Riksbank further develops this model it may include risk premiums that are allowed to vary over time. However, that type of development requires a more advanced estimation procedure and evaluation of the estimated model's properties.

Appendix B. RMSE-based interval from a gamma distribution

THE RIKSBANK'S FORECAST AS AN EXPECTED VALUE

If we regard the Riksbank's forecast as an expected value the gamma distribution's parameters can be computed by solving the following equation system for α_h and β_h

$$E(Y) = \alpha_h \beta_h = \hat{y}^{(h)}$$

$$Std(Y) = \sqrt{\alpha_h}\beta_h = RMSE(h)$$

which gives the solution

$$\alpha_h = \left(\frac{\hat{y}^{(h)}}{RMSE(h)}\right)^2$$

$$\beta_h = \frac{RMSE(h)^2}{\hat{y}^{(h)}}.$$

THE RIKSBANK'S FORECAST AS A MEDIAN

If we instead regard the Riksbank's forecast as a median the forecast errors are $e_t = y_t - Median(y_t \mid y_{t-1}, y_{t-2}, \ldots)$, where $Median(y_t \mid y_{t-1}, y_{t-2}, \ldots)$ is the median in the gamma forecast distribution. To solve for the two parameters of the gamma distribution α_h and β_h we now need expressions for the median in a gamma distribution and for $E(e_t^2)$ (which is matched with $(1/T)\sum_{t=1}^{T} e_t^2) \sim Gamma(\alpha,\beta)$ then the following holds

$$Median(Y) = \beta\Gamma^{-1}(\alpha,1/2)$$

and

$$E[Y - Median(Y)]^2 = \beta^2\{\alpha(1+\alpha) - 2\alpha\Gamma^{-1}(\alpha,1/2) + [\Gamma^{-1}(\alpha,1/2)]^2\}$$

where $\Gamma^{-1}(\alpha,1/2)$ is the inverse of the regularised incomplete gamma function (Abramowitz and Stegun, 1965). If the forecast distribution is $Gamma(\alpha_h,\beta_h)$ and the Riksbank's point forecast is to be regarded as a median then the forecast distribution parameters can be computed by solving the non-linear equation system

$$\beta_h\Gamma^{-1}(\alpha_h,1/2) = \hat{y}^{(h)}$$

$$\beta_h^2\{\alpha_h(1+\alpha_h) - 2\alpha_h\Gamma^{-1}(\alpha_h,1/2) + [\Gamma^{-1}(\alpha_h,1/2)]\} = RMSE(h)$$

for α_h and β_h. If we substitute $\beta_h = \hat{y}^{(h)}/\Gamma^{-1}(\alpha_h,1/2)$ in the second equation we can then solve the equation

$$(\hat{y}^{(h)})^2\left[\frac{\alpha_h(1+\alpha_h)}{(\Gamma^{-1}(\alpha_h,0,1/2))^2} - \frac{2\alpha_h}{\Gamma^{-1}(\alpha_h,0,1/2)} + 1\right] = RMSE(h)$$

numerically for α_h using for example Newton's method. The solution for β_h is now given by $\beta_h = \hat{y}^{(h)}/\Gamma^{-1}(\alpha_h,1/2)$.

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