Regime Switching GARCH Models

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Stockholm, September 2006

Motivation-Interpretation

- Last 20 years: lot of research on modelling volatility in financial markets using GARCH type models.

- Simple GARCH models: usually (too) highly persistent.

- Source of misspecification: the structural form of conditional mean and variance is relatively inflexible and held fixed throughout the entire sample.

Literature

- Schwert (1989): model where returns switch between high or low variance states according to a two state Markov process.

- Hamilton and Susmel (1994) and Cai (1994): ARCH model with regime switching parameters.

- Gray (1996): proposes a class of regime-switching GARCH (RS-GARCH) models with time-varying probability, but estimates an approximation to the model. The quality of the approximation is not known.

  See also Dueker (1997), Bollen et al. (2000), Klaassen (2002), and Haas et al. (2004).

Our contribution

- Sufficient conditions for geometric ergodicity, stationarity, and existence of moments of a class of RS-GARCH models (à la Gray) with time-varying (or constant) probability.

  NB: Zhang, Li, and Yuen (JTSA, July 2006) study the constant probability RS-GARCH model. They provide results on stationarity and tail behaviour for K components with P and Q lags and moments for $K = 2$ and $P = Q = 1$. They do not take account of path dependence in estimation.

- Bayesian estimation (Gibbs sampling) of the RS-GARCH model tackling properly path dependence by data augmentation.

- Simulation example and application to NASDAQ daily returns.

Outline

1. RS-GARCH

2. Inference

3. Application

4. Conclusion

RS-GARCH: Definition

- RS-GARCH model:

$$y_t = \mu_{s_t} + u_t = \mu_{s_t} + \epsilon_t \sigma_t(s_t), \quad \epsilon_t \sim IID(0,1)$$

$$\sigma_t^2(s_t) = \omega_{s_t} + \beta_{s_t} \sigma_{t-1}^2(s_{t-1}) + \alpha_{s_t} u_{t-1}^2$$

- s_t: discrete random variable with support $\{1, 2, \ldots, n\}$

  $\longrightarrow$ indicates the active parameters at date t.

- Model must be closed by specifying the distribution of s_t, and the relation between s_t, $t = 1, 2, \ldots, T$.

RS-GARCH: State probabilities

- **Markov-switching**: s_t follows a Markov chain with fixed probabilities

  $\rightarrow$ e.g. two states, two transition probabilities.

- **Time-varying probabilities**: s_t conditionally independent,

$$s_t \sim \text{Multinomial}(p_{jt}, j = 1, \ldots, n)$$
$$p_{jt} = \Pr(s_t = j | \Im_{t-1}) = p_{jt}(\Im_{t-1})$$

where $\Im_t = (y_t \ldots, y_1)$.

RS-GARCH: State probabilities

- For 2 regimes, we take

$$p_{1t} = \left[1 + \exp(\delta_0 + \delta_1 y_{t-1}^2)\right]^{-1}$$

with $\delta_1 < 0$, thus, $p_{1t} \to 1$ as $y_{t-1}^2 \to \infty$.

- Probabilities of non-stable regimes can be "high" in tranquil periods

  $\rightarrow$ explains the substantial growth in volatility in a short amount of time.

- Large shocks have the effect of “relieving pressure” by reducing the probability of a large shock in the next period.

RS-GARCH: Theory

- RS-GARCH model is stationary and ergodic if

  (A1) $\beta_1 + \alpha_1 < 1$

  (A2) $\beta_j > 0, \;\; \alpha_j > 0$ for $j = 1, 2, \ldots n$

  (A3) $0 < p_{1t}(y_{t-1}^2)$ and $p_{1t}(y_{t-1}^2) \to 1$ as $y_{t-1}^2 \to \infty$ for all t.

  $\Rightarrow$ One regime must be stable, but the other regimes can be explosive.

- Given A2 and A3, moments of order $2k$ exist if $\mathsf{E}(u_t^{2k}) < \infty$ and $\mathsf{E}(\beta_1 + \gamma_1 u_t^2)^k < 1$.

Inference

- Joint density of $y = (y_1, y_2, \ldots, y_T)$ and $S = (s_1, s_2, \ldots, s_T)$ given the parameters:

$$f(y, S|\mu, \theta, \delta) \propto \prod_{t=1}^{T} \sigma_t^{-1} \exp\left(-\frac{(y_t - \mu_{s_t})^2}{2\sigma_t^2}\right) p_{1t}^{2-s_t}(1-p_{1t})^{s_t-1}$$

  Parameters: $\delta = (\delta_0, \delta_1)$, $\mu = (\mu_1, \mu_2)$, and $\theta = (\theta_1', \theta_2')'$, where $\theta_k' = (\omega_k, \beta_k, \alpha_k)$ for $k = 1, 2$.

- S is unobservable $\Rightarrow$ integrate out all possible paths to compute the likelihood of y

  $\Rightarrow$ computationally unfeasible because of the path-dependence problem: GARCH conditional variance at time t depends on the entire sequence of states up to time $t-1$ (2^{t-1} paths).

Path-dependence with two states

$$\sigma^2_{2|1,1} = \omega_1 + \beta_1 \sigma^2_{1|1} + \alpha_1 u_1^2$$

$$\sigma^2_{1|1} = \omega_1 + \beta_1 \sigma^2_0 + \alpha_1 u_0^2$$

$$\sigma^2_{2|1,2} = \omega_2 + \beta_2 \sigma^2_{1|1} + \alpha_2 u_1^2$$

$$\sigma^2_0, u_0$$

$$\sigma^2_{2|2,1} = \omega_1 + \beta_1 \sigma^2_{1|2} + \alpha_1 u_1^2$$

$$\sigma^2_{1|2} = \omega_2 + \beta_2 \sigma^2_0 + \alpha_2 u_0^2$$

$$\sigma^2_{2|2,2} = \omega_2 + \beta_2 \sigma^2_{1|2} + \alpha_2 u_1^2$$

Bayesian Inference

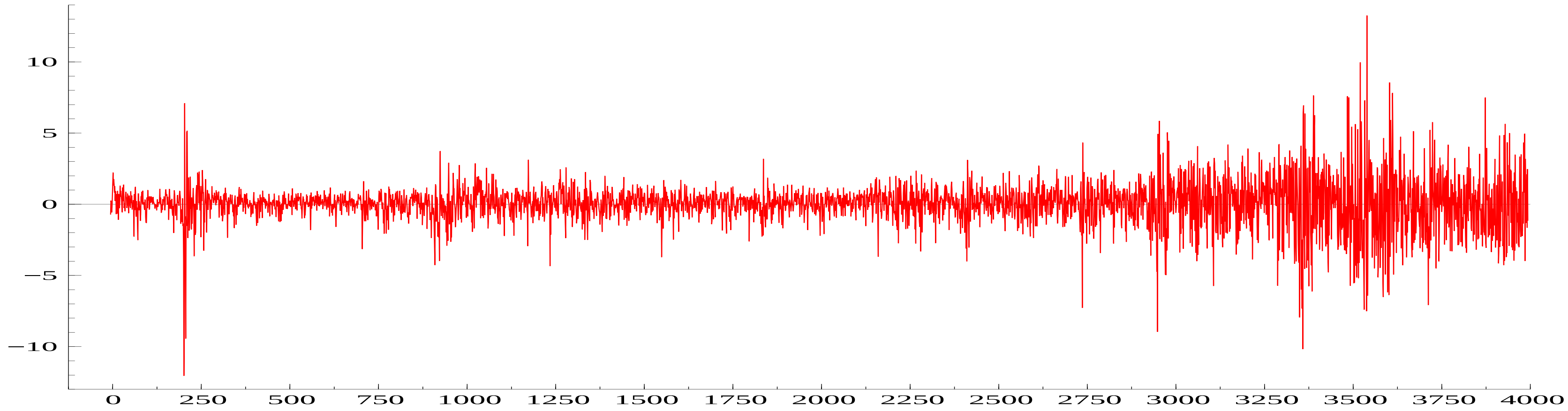
Gibbs sampling in 4 blocks:

1. $s_t | S_{>t}, \mu, \theta, \delta, y \qquad t = 1, \ldots, T$ (single move)
2. $\theta | S, \mu, y$ (griddy Gibbs)
3. $\mu | S, \theta, y$ (griddy Gibbs)
4. $\delta | S, y$ (griddy Gibbs)

Application

NASDAQ daily returns from 23/12/1986 to 28/10/2002 (4,000 observations)

Mean	0.033	Minimum	-12.05
Standard deviation	1.52	Maximum	13.26
Skewness	-0.24	Kurtosis	10.93



Posterior means and standard deviations

	RS-GARCH	MN-GARCH	N-GARCH	t-GARCH
ω_1	0.0073 (0.0052)	0.0008 ((0.0007)	0.022 (0.0033)	0.0073 (0.0028)
β_1	0.91 (0.038)	0.92 ((0.010)	0.85 (0.012)	0.89 (0.020)
α_1	0.012 (0.0086)	0.057 ((0.0087)	0.15 (0.013)	0.082 (0.016)
ω_2	0.11 (0.070)	0.12 ((0.049)	-	-
β_2	1.45 (0.32)	0.78 ((0.157)	-	-
α_2	0.43 (0.24)	0.63 ((0.055)	-	-
μ_1	0.11 (0.014)	0.17 ((0.018)	0.082 (0.0098)	0.11 (0.013)
μ_2	-0.005 (0.079)	-0.64 ((0.118)	-	-
δ_0	-2.22 (0.45)	p : 0.85 ((0.029)	-	-
δ_1	-0.032 (0.027)	-	-	-
ν	-	-	-	7.09 (0.91)

MN-GARCH model of Haas et al. (2004)

$$
\begin{aligned}
y_t &= \begin{cases} \mu_1 + \sigma_{1t}\epsilon_t, & \text{with probability } p \\ \mu_2 + \sigma_{2t}\epsilon_t, & \text{with probability } 1-p, \end{cases} && (1)\\
\sigma^2_{1t} &= \omega_1 + \beta_1 \sigma^2_{1,t-1} + \alpha_1 (y_{t-1} - \mu_1)^2 && (2)\\
\sigma^2_{2t} &= \omega_2 + \beta_2 \sigma^2_{1,t-1} + \alpha_2 (y_{t-1} - \mu_2)^2 && (3)\\
\epsilon_t &\sim i.i.d.\ N(0,1). && (4)
\end{aligned}
$$

No path dependence (two parallel variance processes).

Bayesian inference by Bauwens and Rombouts (2006).

Application

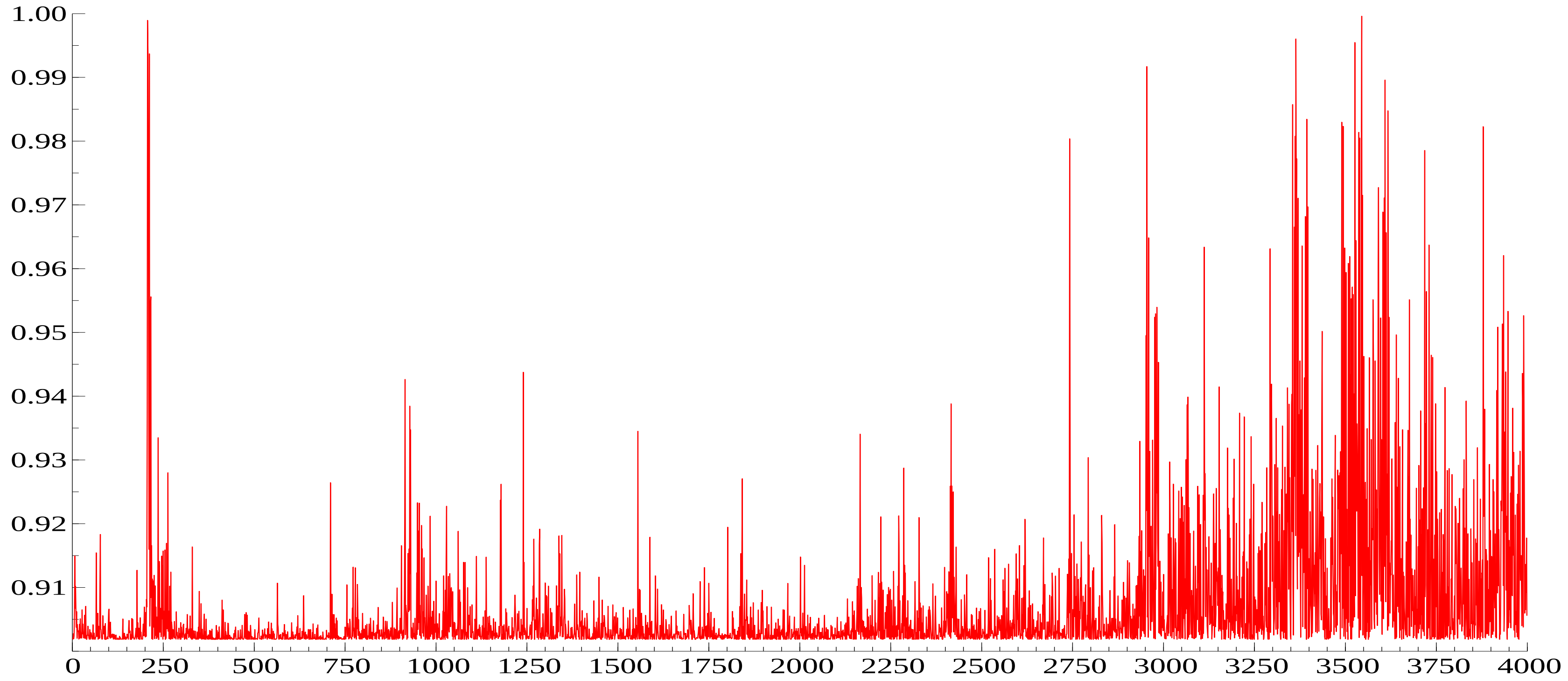
- Stationarity conditions hold for the estimated RS-GARCH model without being imposed.
- The estimated RS- and MN-GARCH models fit the empirical moments of the data differently:

	Empirical	RS-GARCH	MN-GARCH
Variance	2.31	2.28	2.06
Skewness	-0.24	-0.01	-0.30
Kurtosis	10.9	12.7	18.9

- The normal and t-GARCH estimated models imply an infinite unconditional variance with high probability.
- The MN-GARCH model fits the autocorrelations slightly better than the RS-GARCH model.

Probabilities of stable regime

The probability of the stable regime fluctuates between 0.90 and 0.999.



Conclusions

- Estimation of RS-GARCH by Bayesian inference feasible.

- Future research:

  Improved algorithms

  Other applications

  Forecast evaluation

  Other specifications (e.g. of probability).

- Extension to multivariate and ACD models in progress.