Estimating Macroeconomic Models:
A Likelihood Approach

Jesús Fernández-Villaverde
Duke University, NBER, and CEPR

Juan Rubio-Ramírez
Duke University

Outline

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- We apply particle filtering to evaluate the likelihood of the model.
- We estimate a neoclassical business cycle model with investment-specific technological change and stochastic volatility.

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- We want to track conditional density $p\left(S_t | y^{t-1}; \gamma\right)$.

Factorization of the Likelihood

- Why?

$$
\begin{aligned}
p\left(y^{T};\gamma\right) &= \prod_{t=1}^{T} p\left(y_t|y^{t-1};\gamma\right) \\
&= \prod_{t=1}^{T} \int p\left(y_t|S_t;\gamma\right) p\left(S_t|y^{t-1};\gamma\right) dS_t
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- Knowledge of $\left\{p\left(S_t|y^{t-1};\gamma\right)\right\}_{t=1}^{T}$ allows the evaluation of the likelihood of the model.

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1. Forecast: Chapman-Kolmogorov equation

$$p\left(S_t|y^{t-1};\gamma\right) = \int p\left(S_t|S_{t-1};\gamma\right) p\left(S_{t-1}|y^{t-1};\gamma\right) dS_{t-1}$$

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2. Update: Bayes' theorem

$$p\left(S_t|y^{t};\gamma\right) = \frac{p\left(y_t|S_t;\gamma\right) p\left(S_t|y^{t-1};\gamma\right)}{p\left(y_t|y^{t-1};\gamma\right)}$$

where:

$$p\left(y_t|y^{t-1};\gamma\right) = \int p\left(y_t|S_t;\gamma\right) p\left(S_t|y^{t-1};\gamma\right) dS_t$$

A Law of Large Numbers

$$\prod_{t=1}^{T} \int p\left(y_t|S_t;\gamma\right) p\left(S_t|y^{t-1};\gamma\right) dS_t$$

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Suppose we have $\left\{ \left\{ s_{t|t-1}^{i} \right\}_{i=1}^{N} \right\}_{t=1}^{T} \sim \left\{ p\left(S_t | y^{t-1}; \gamma\right) \right\}_{t=1}^{T}$.

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Then:

$$p\left(y^T; \gamma\right) \simeq \prod_{t=1}^{T} \frac{1}{N} \sum_{i=1}^{N} p\left(y_t | s_{t|t-1}^{i}; \gamma\right)$$

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Evaluating the likelihood function $\Leftrightarrow$ Drawing from density:

$$\left\{ p\left(S_t | y^{t-1}; \gamma\right) \right\}_{t=1}^{T}$$

A Proposition

- Let $\left\{ s^{i}_{t|t-1} \right\}_{i=1}^{N}$ be a draw from $p\left(S_t|y^{t-1};\gamma\right)$.

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- Let $\{\widetilde{s}^i_t\}_{i=1}^N$ be a draw with replacement from $\left\{s^i_{t|t-1}\right\}_{i=1}^N$ and probabilities:

$$q^i_t = \frac{p\left(y_t|s^i_{t|t-1};\gamma\right)}{\sum_{i=1}^N p\left(y_t|s^i_{t|t-1};\gamma\right)}$$

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- Then, $\{\widetilde{s}_t^i\}_{i=1}^{N}$ is a draw from $p\left(S_t|y^t;\gamma\right)$:

$$\left\{s_{t|t}^{i}\right\}_{i=1}^{N} = \{\widetilde{s}_t^i\}_{i=1}^{N}$$

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- Let $\left\{s^i_{t|t-1}\right\}_{i=1}^N$ be a draw from $p\left(S_t|y^{t-1};\gamma\right)$.
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- Then, $\{\widetilde{s}^i_t\}_{i=1}^N$ is a draw from $p\left(S_t|y^t;\gamma\right)$:

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- Proof: Importance sampling and Bayes' theorem.

Importance of the Proposition

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1. Update: We can use a draw $\left\{s_{t|t-1}^{i}\right\}_{i=1}^{N}$ from $p\left(S_t|y^{t-1};\gamma\right)$ to get a draw $\left\{s_{t|t}^{i}\right\}_{i=1}^{N}$ from $p\left(S_t|y^{t};\gamma\right)$.

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2. Forecast: We can use a draw $\left\{s^i_{t|t}\right\}_{i=1}^N$ from $p\left(S_t|y^t;\gamma\right)$, a draw from $p\left(W_{t+1};\gamma\right)$, and $S_{t+1}=f\left(S_t,W_{t+1};\gamma\right)$ to get a draw $\left\{s^i_{t+1|t}\right\}_{i=1}^N$.

Particle Filtering I

Step 0, Initialization: Sample N values $\left\{s^i_{1|0}\right\}_{i=1}^N$ from $p\left(S_1;\gamma\right)$. Go to step 2.

Step 1, Forecast: Sample N values $\left\{s^i_{t|t-1}\right\}_{i=1}^N$ with $p\left(W_t;\gamma\right)$ and $S_t = f\left(S_{t-1}, W_t;\gamma\right)$.

Step 2, Weighting: Assign to each draw $s^i_{t|t-1}$ the weight q^i_t.

Step 3, Update: Draw $\left\{s^i_{t|t}\right\}_{i=1}^N$ with replacement from $\left\{s^i_{t|t-1}\right\}_{i=1}^N$ with probabilities $\left\{q^i_t\right\}_{i=1}^N$. If $t < T$ set $t \rightsquigarrow t+1$ and go to step 1. Otherwise stop.

Particle Filtering II

Use $\left\{\left\{s_{t|t-1}^{i}\right\}_{i=1}^{N}\right\}_{t=1}^{T}$ to compute:

$$p\left(y^{T};\gamma\right)\simeq\prod_{t=1}^{T}\frac{1}{N}\sum_{i=1}^{N}p\left(y_{t}|s_{t|t-1}^{i};\gamma\right)$$

We can filter, forecast, and smooth

Numerical accuracy: effective sample size, density ratio test, etc.

An Application: a Business Cycle Model

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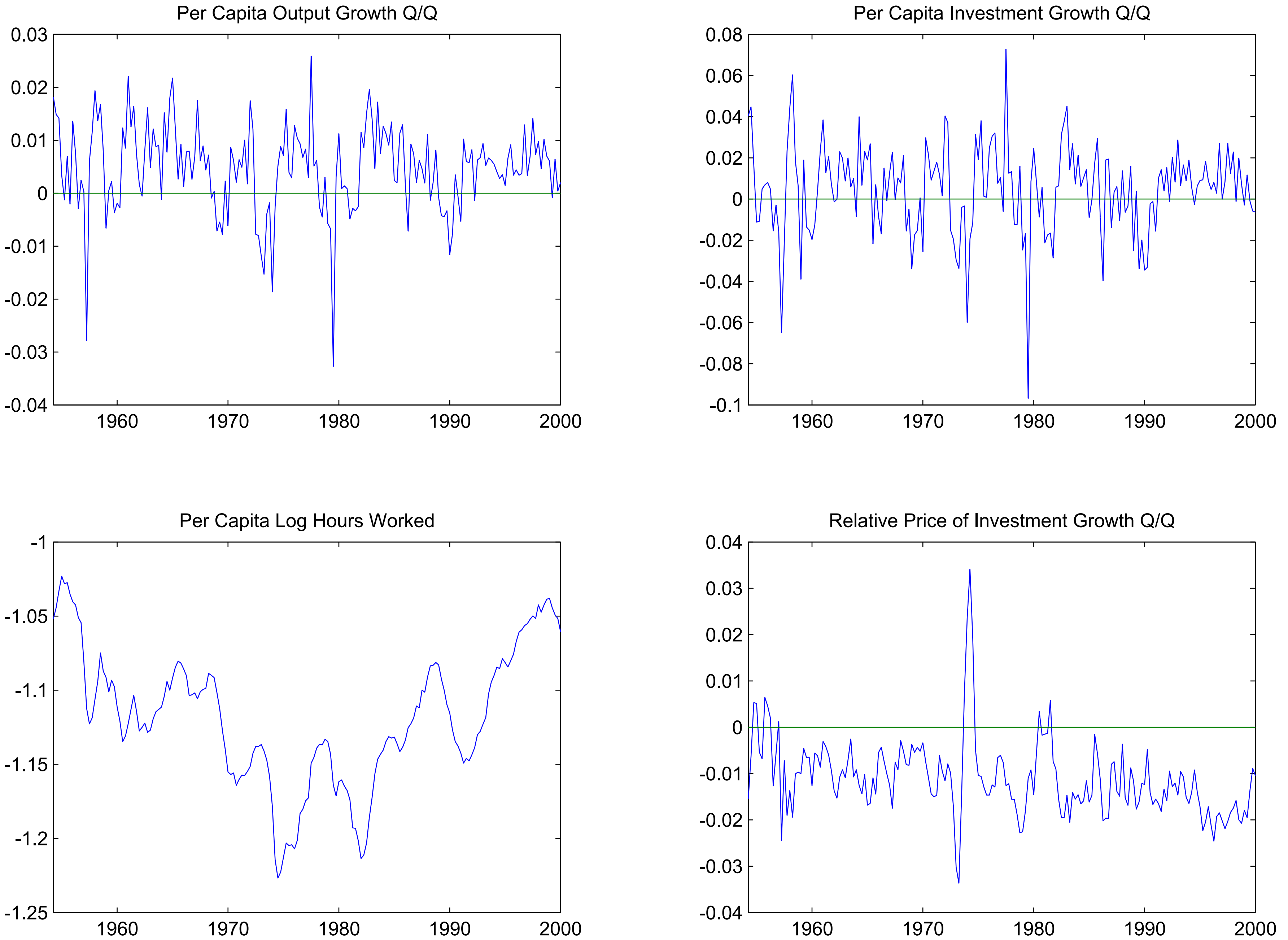
- And learn something about the evolution of the U.S. aggregate fluctuations over the last decades.
- A business cycle model with:

1. Investment-specific technological change. Greenwood, Herkowitz, and Krusell (1997 and 2000)
2. Stochastic volatility.


Per Capita Output Growth Q/Q
Per Capita Investment Growth Q/Q
Per Capita Log Hours Worked
Relative Price of Investment Growth Q/Q
0.03
0.02
0.01
0
-0.01
-0.02
-0.03
-0.04
0.08
0.06
0.04
0.02
-0.02
-0.04
-0.06
-0.08
-0.1
-1
-1.05
-1.1
-1.15
-1.2
-1.25
1960
1970
1980
1990
2000

Environment

- Representative household with utility function:

$$E_0 \sum_{t=0}^{\infty} \beta^t \left(e^{d_t} \log C_t + \psi \log (1 - L_t) \right)$$

- Final Good: $C_t + X_t = A_t K_t^{\alpha} L_t^{1-\alpha}$
- Law of motion of capital: $K_{t+1} = (1 - \delta) K_t + V_t X_t$
- Shocks:

$$\begin{aligned} d_t &= \rho d_{t-1} + \sigma_{dt} \varepsilon_{dt}, \ \varepsilon_{dt} \sim \mathcal{N}(0,1) \\ \log A_t &= \zeta + \log A_{t-1} + \sigma_{at} \varepsilon_{at}, \ \gamma \geq 0 \text{ and } \varepsilon_{at} \sim \mathcal{N}(0,1) \\ \log V_t &= \upsilon + \log V_{t-1} + \sigma_{\upsilon t} \varepsilon_{\upsilon t}, \ \upsilon \geq 0 \text{ and } \varepsilon_{\upsilon t} \sim \mathcal{N}(0,1) \end{aligned}$$

Stochastic Volatility

We follow a standard specification:

$$\log \sigma_{dt} = (1 - \lambda_d) \log \overline{\sigma}_d + \lambda_d \log \sigma_{dt-1} + \tau_d \eta_{dt} \text{ and } \eta_{dt} \sim \mathcal{N}(0,1)$$

$$\log \sigma_{at} = (1 - \lambda_a) \log \overline{\sigma}_a + \lambda_a \log \sigma_{at-1} + \tau_a \eta_{at} \text{ and } \eta_{at} \sim \mathcal{N}(0,1)$$

$$\log \sigma_{\upsilon t} = (1 - \lambda_\upsilon) \log \overline{\sigma}_\upsilon + \lambda_\upsilon \log \sigma_{\upsilon t-1} + \tau_\upsilon \eta_{\upsilon t} \text{ and } \eta_{\upsilon t} \sim \mathcal{N}(0,1)$$

Performing Likelihood-Based Inference

- We compute the model using a perturbation method.
- Time series:

  1. Relative price of capital, output, investment, and hours.
  2. Sample: 1955:Q1 to 2000:Q4.

- Vector of parameters γ is:

  $$(\rho, \beta, \psi, \alpha, \delta, \upsilon, \zeta, \tau_d, \tau_a, \tau_\upsilon, \overline{\sigma}_d, \overline{\sigma}_a, \overline{\sigma}_\upsilon, \lambda_a, \lambda_\upsilon, \lambda_d, \sigma_1^\epsilon, \sigma_2^\epsilon, \sigma_3^\epsilon)$$

- Use a Random-walk Metropolis-Hastings to explore the likelihood: Classical and Bayesian.

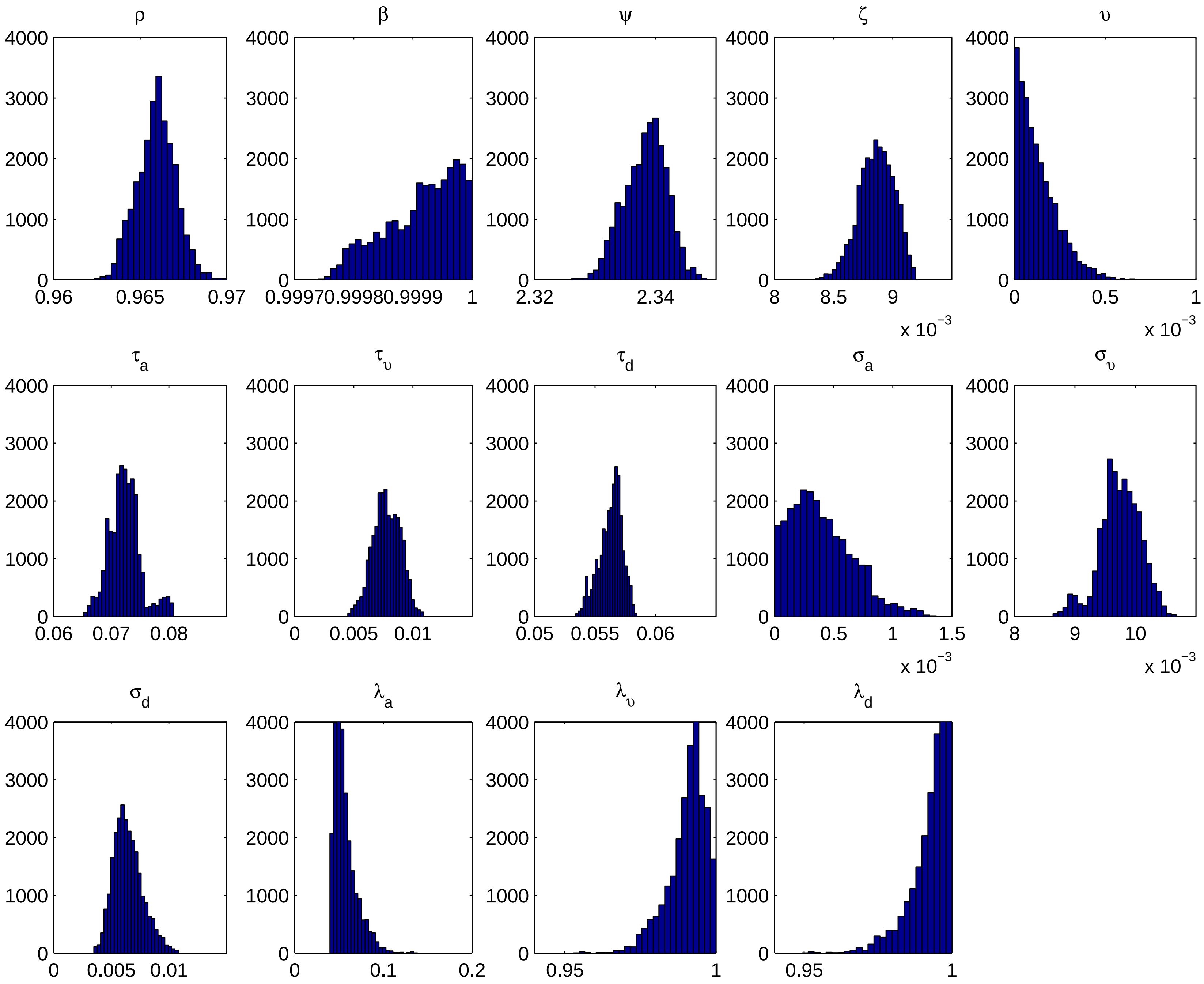

ρ
β
ψ
ζ
υ
τa
τυ
τd
σa
συ
σd
λa
λυ
λd
x 10^−3

Figure 6.1: Model versus Data

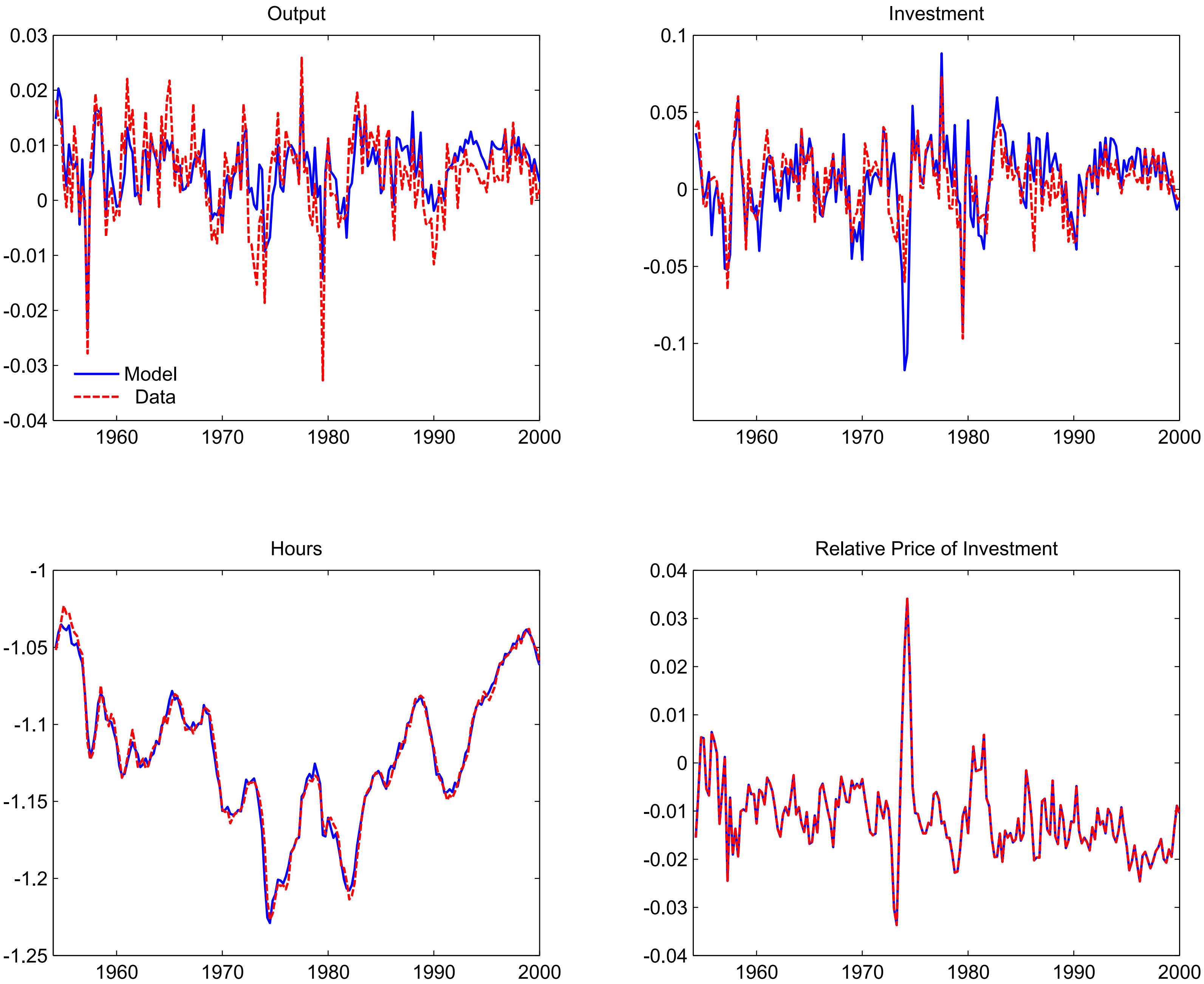


Figure 6.2: Smoothed Capital and Shocks

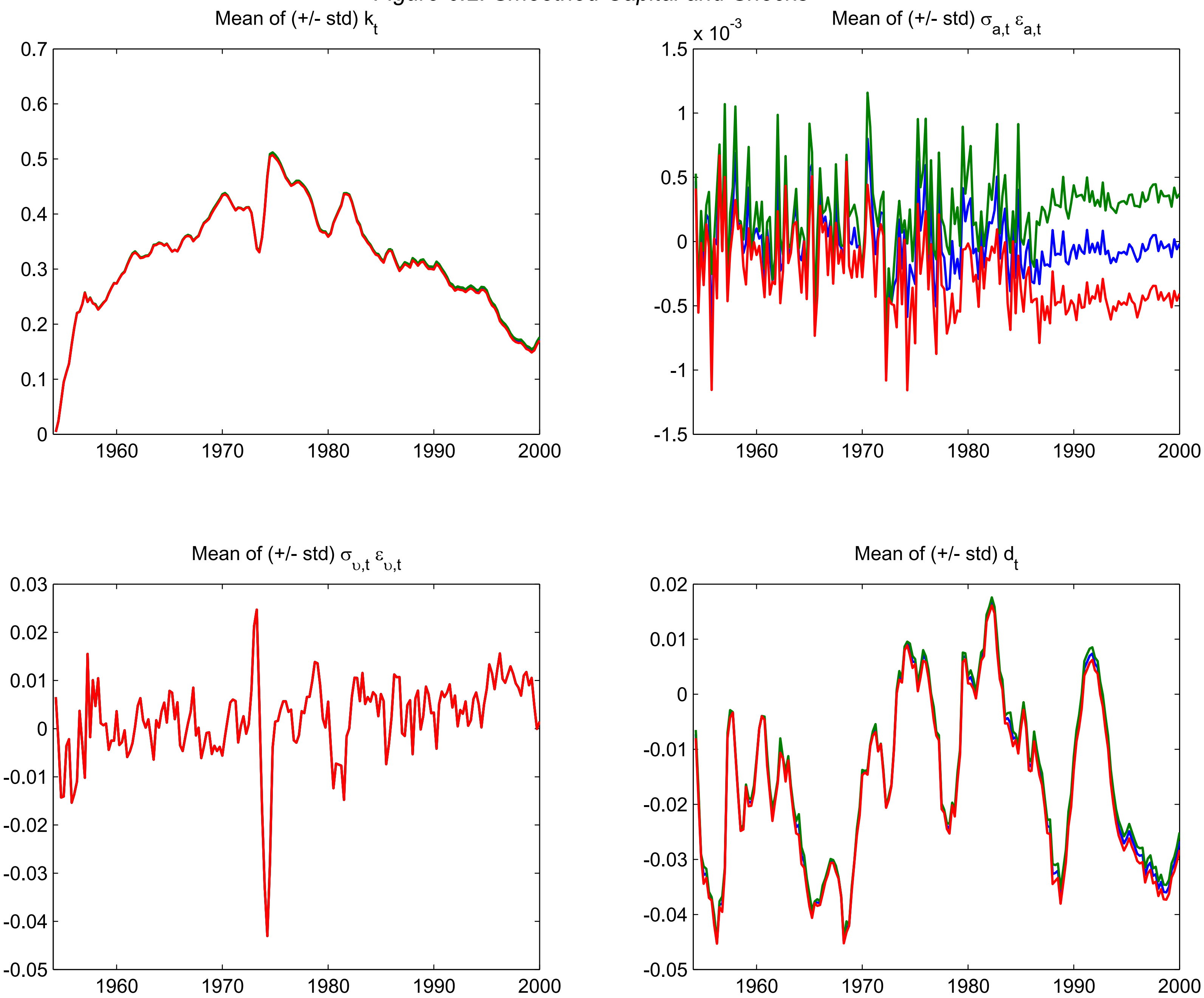


Figure 6.3: Smoothed Volatilities

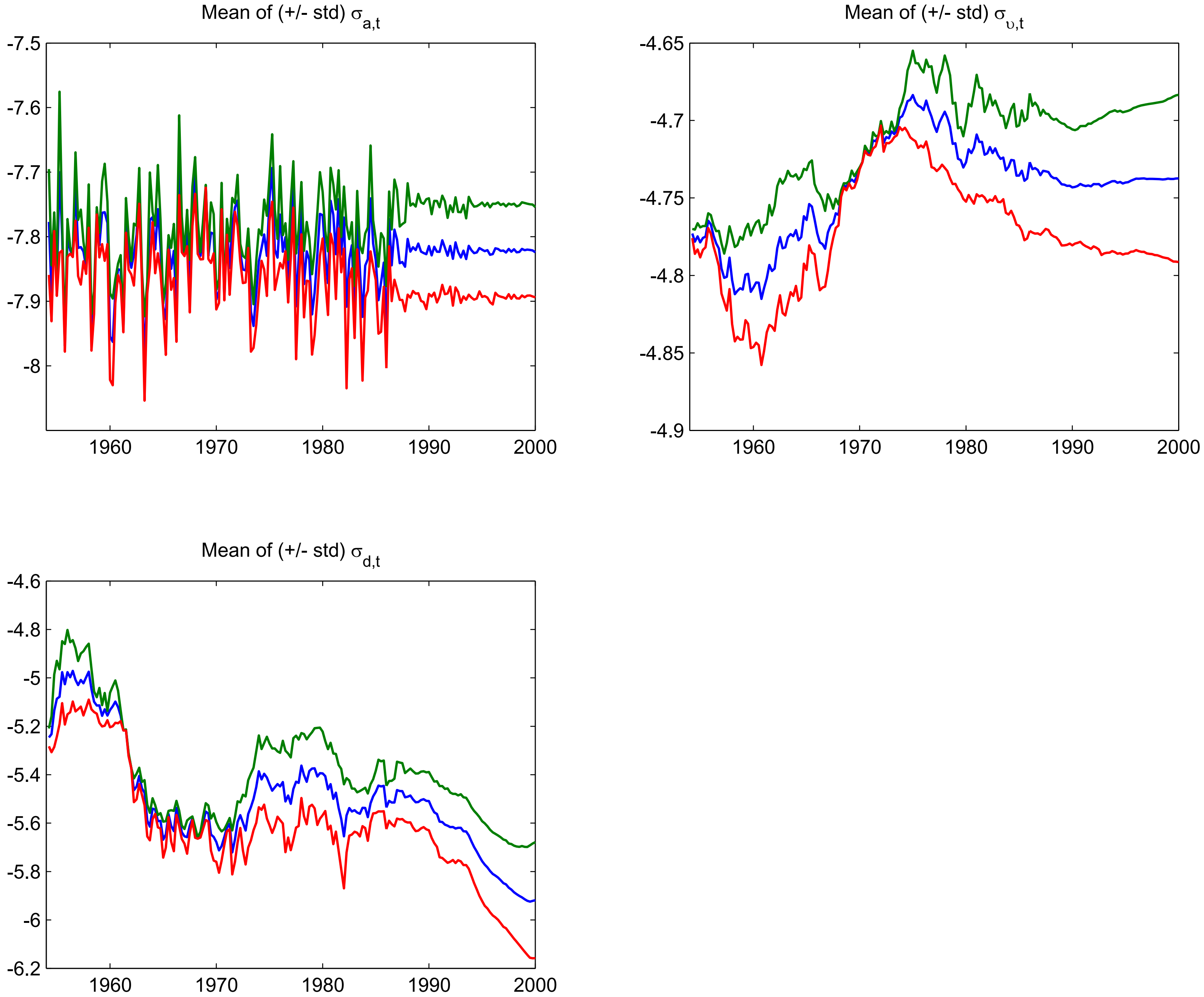


Figure 6.4: Instantaneous Standard Deviation

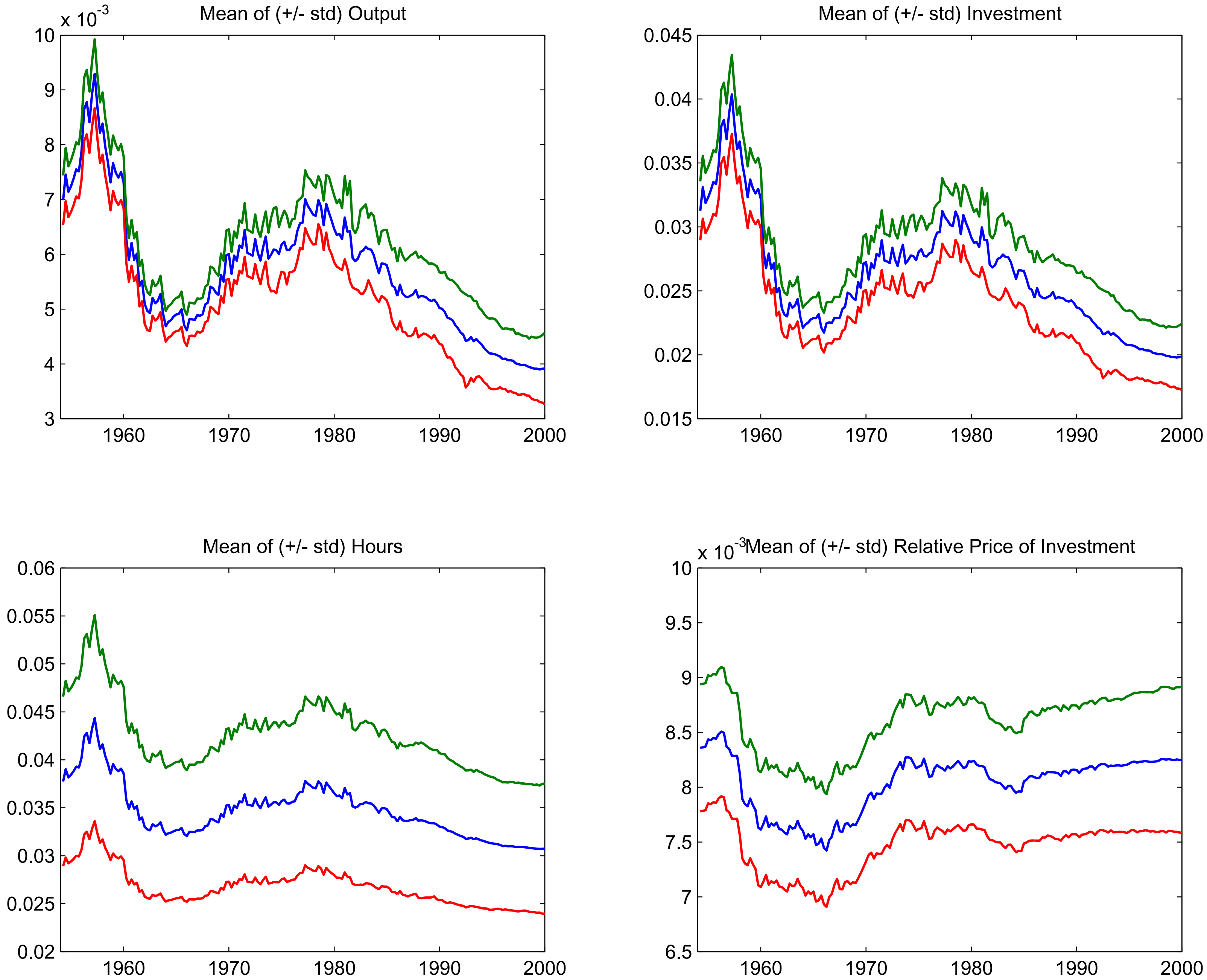


Figure 6.5: Counterfactual Exercise 1

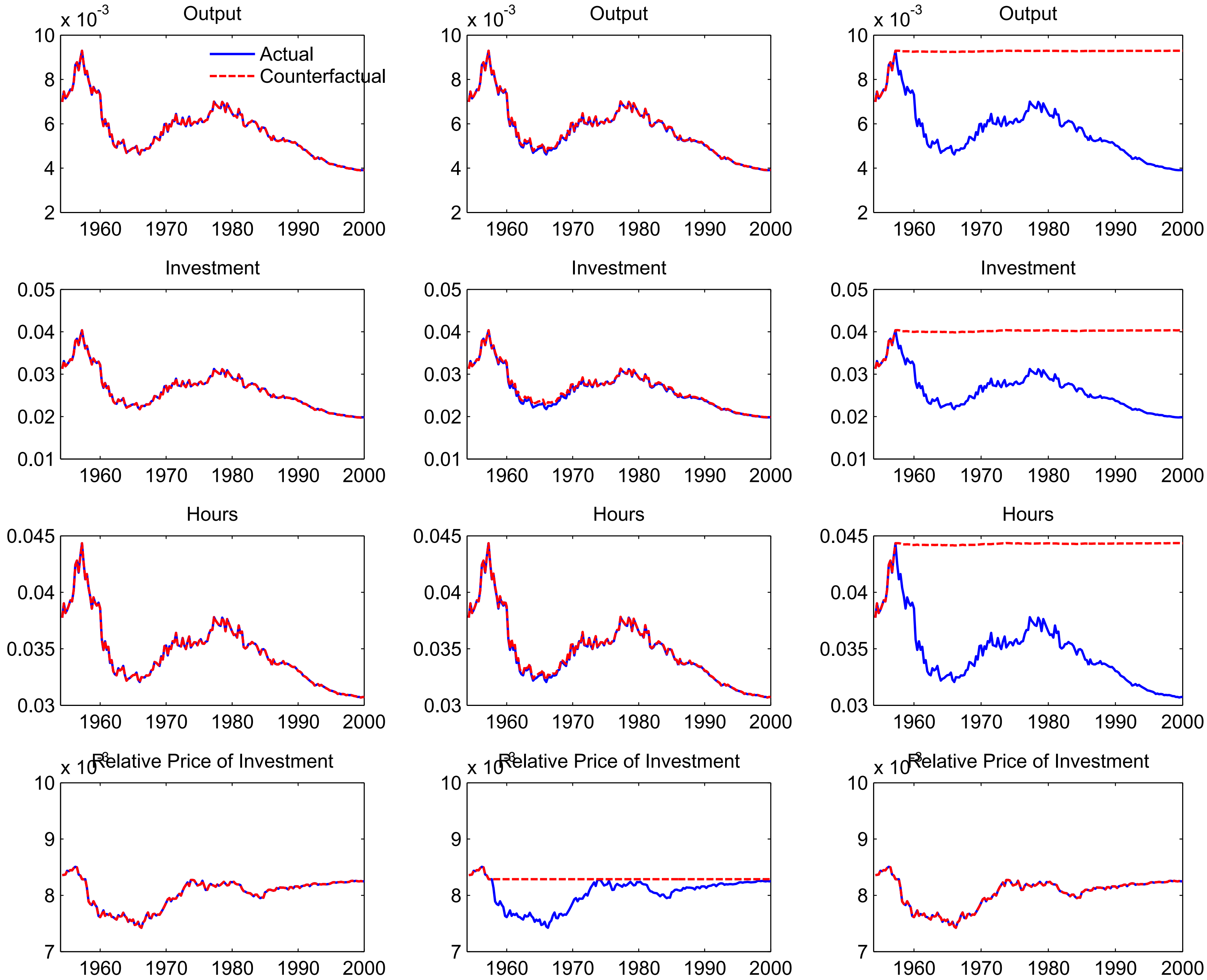


Figure 6.6: Counterfactual Exercise 2

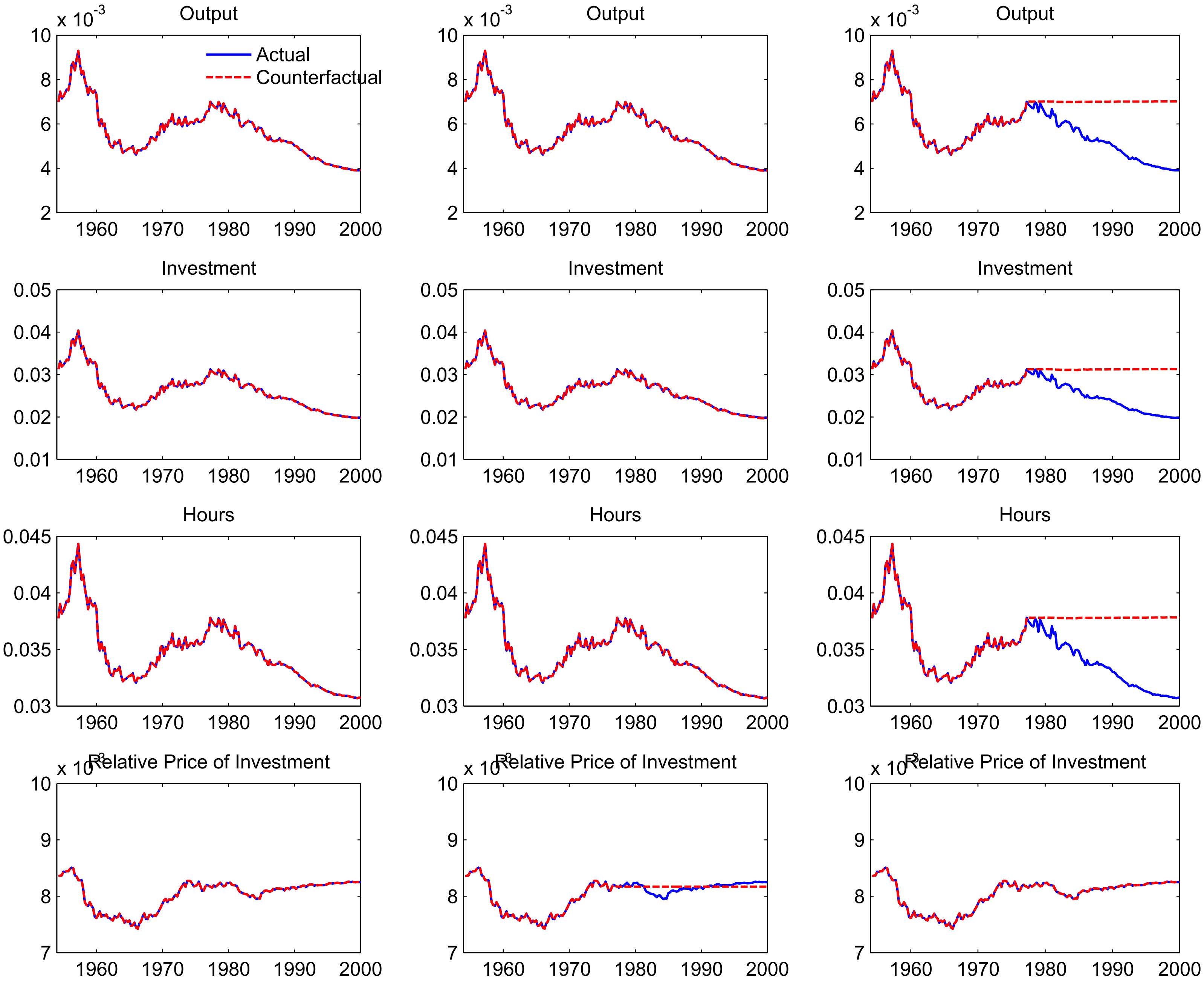
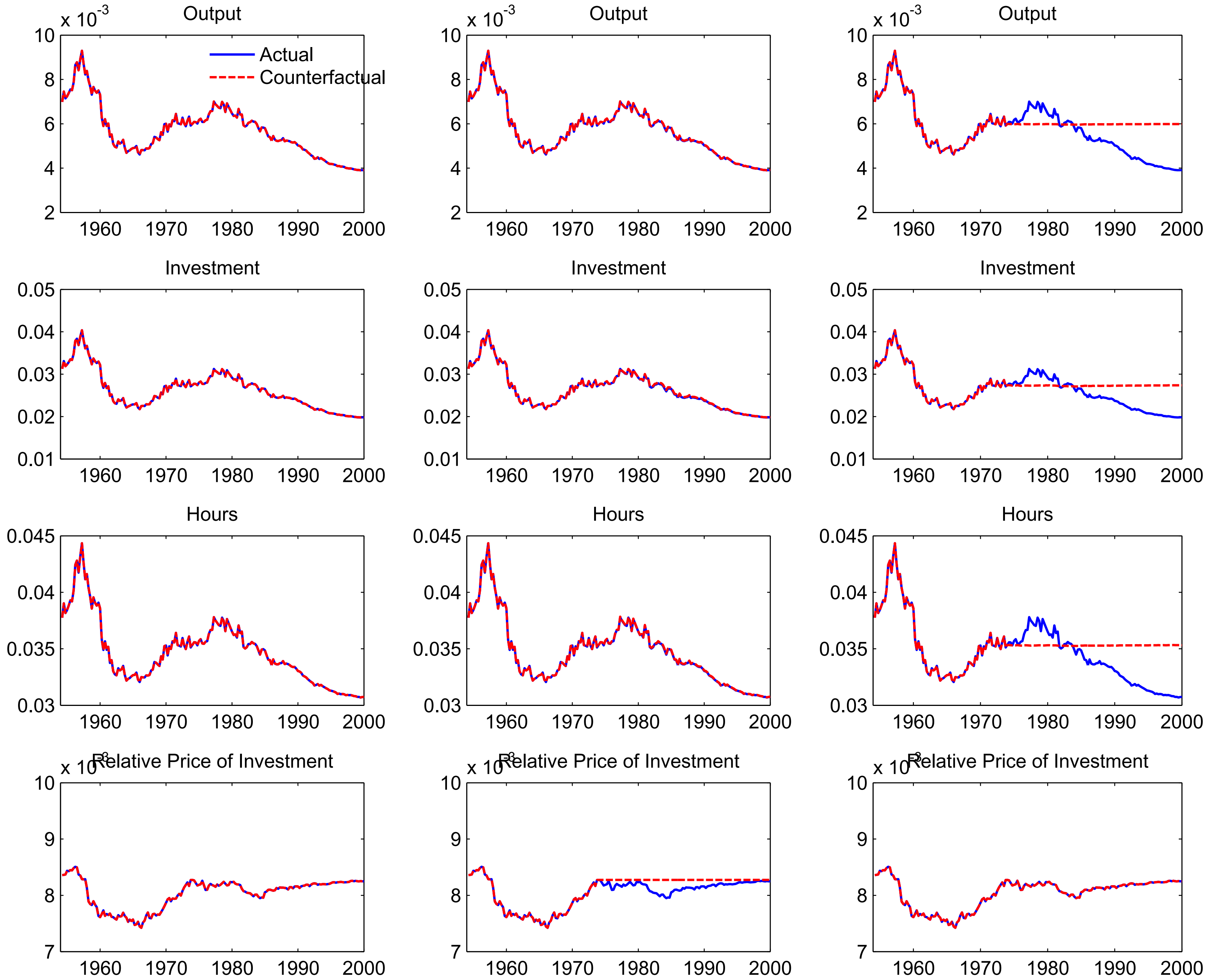


Figure 6.7: Counterfactual Exercise 3



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- We estimate four version of the model:

Table 7.1: Versions of the Model

Solution	No Stochastic Volatility	Stochastic Volatility
Linear	Version 1	Version 2
Quadratic	Version 3	Benchmark

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- Loglike benchmark: 2350.6, loglike version 2: 2230.4

Figure 7.1: Comparison of Smoothed Capital and Shocks

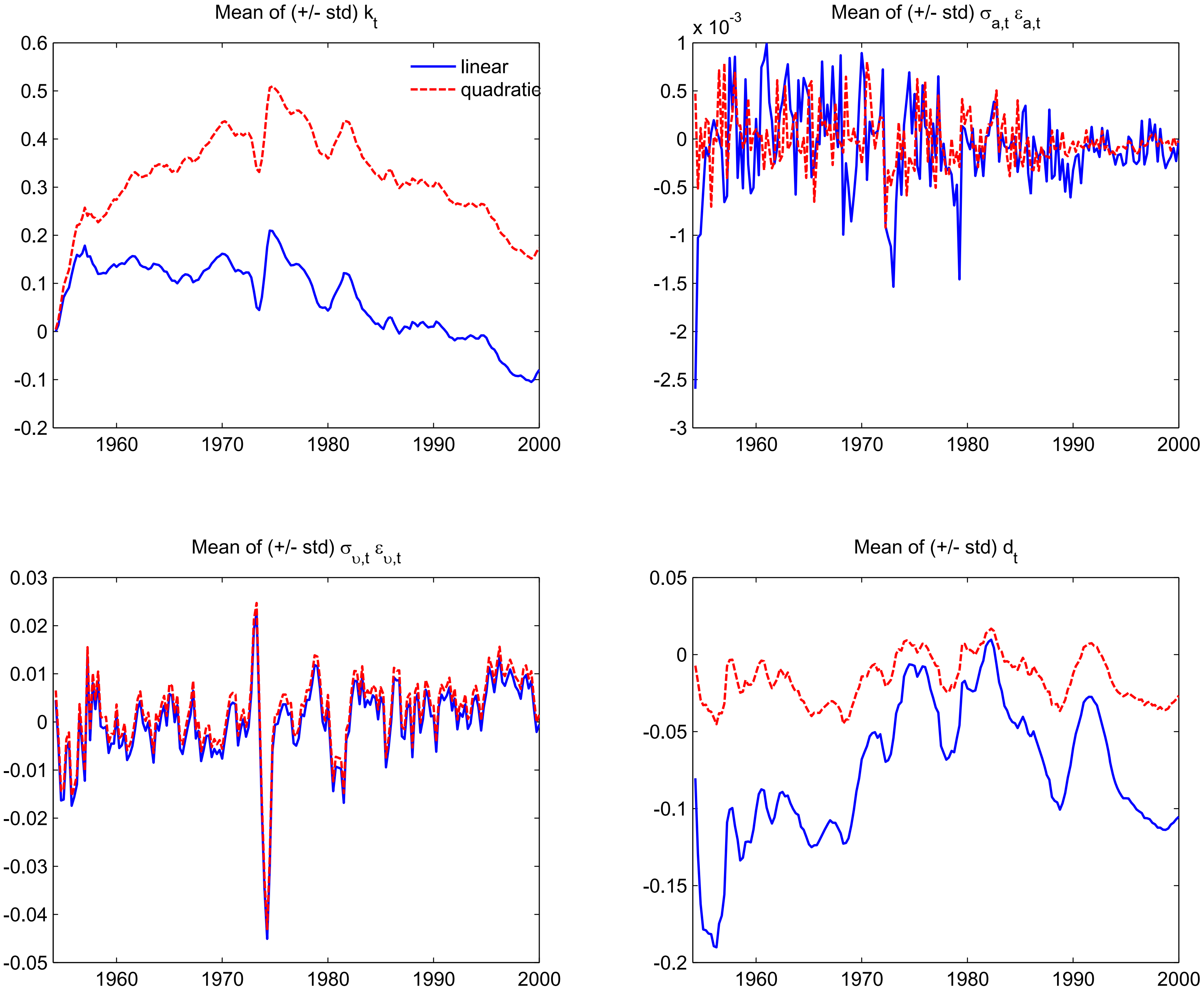
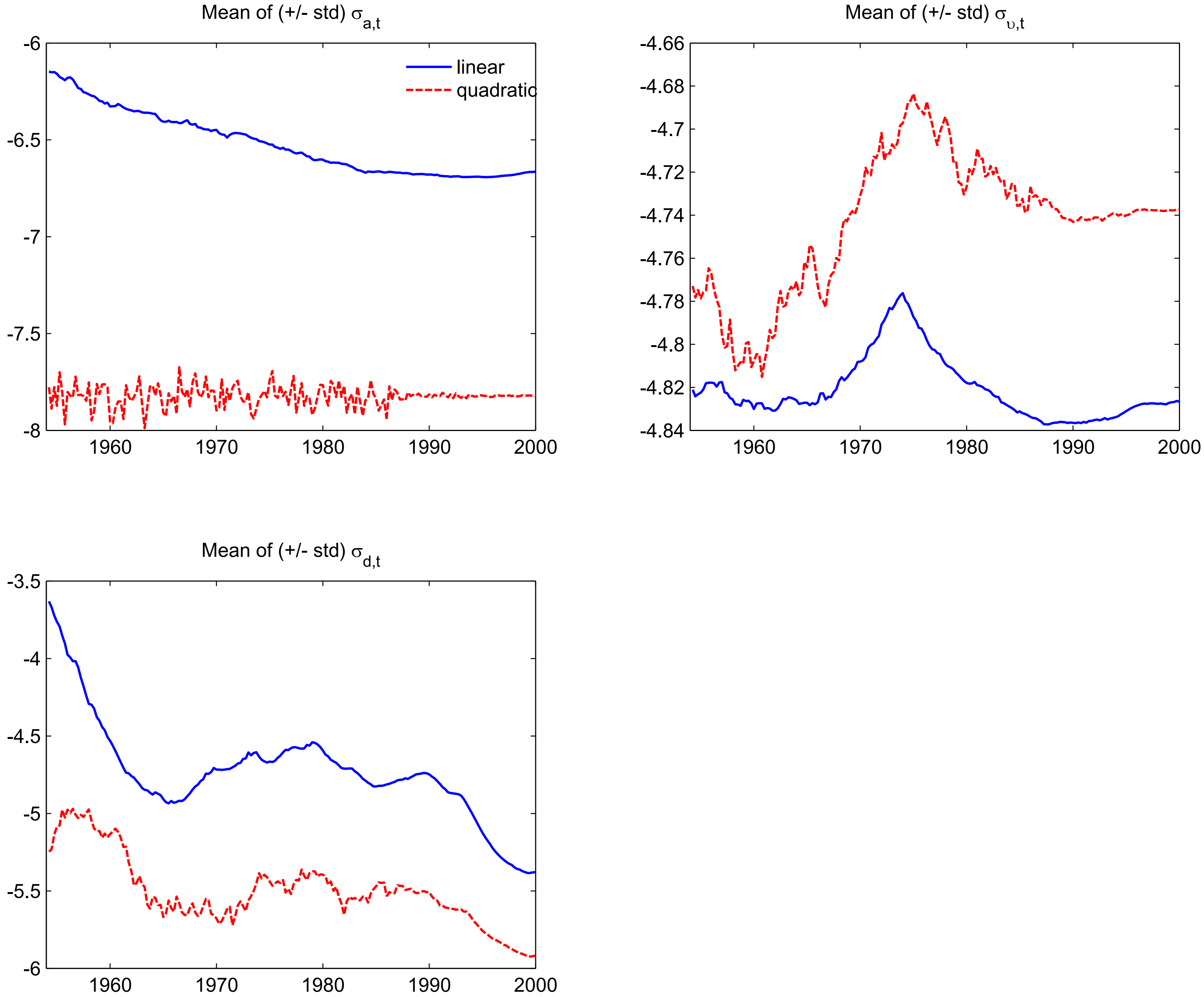


Figure 7.2: Comparison of Smoothed Volatilities



Conclusions

1. Particle filtering is a general purpose and efficient method to estimate DSGE models.

2. We learned about the importance of stochastic volatility to account for U.S. Business Cycle.

3. Much exciting work to do in the next few years!