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Renovatio Monetae: Gesell Taxes in Practice*

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Abstract

Gesell taxes on money holdings have received attention in recent decades as a way of alleviating the zero lower bound on interest rates. Less known is that such a tax was the predominant method used to generate seigniorage in large parts of medieval Europe for around two centuries. When the Gesell tax was levied, current coins ceased to be legal tender and had to be exchanged into new coins for a fee - an institution known as renovatio monetae or periodic re-coinage. This could occur as often as twice a year. Using a cash-in-advance model, we analyze under which conditions agents prefer to re-mint their coins and the system generates tax revenues. We also analyze how prices fluctuate over an issue period.

Keywords: Seigniorage, Gesell tax, periodic re-coinage, cash-in-advance model

JEL classification: E31, E42, E52, N13.

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†The Research Institute of Industrial Economics (IFN), P.O. Box 55665, SE-10215 Stockholm, Sweden. Correspondence: roger.svensson@ifn.se

‡Research Department, Sveriges Riksbank and Uppsala Center for Labor Studies, SE-103 37, Stockholm, Sweden. Correspondence: andreas.westermark@riksbank.se.

1 Introduction

First proposed by Gesell (1906), the idea of a tax on money holdings has received increasing attention in recent decades. The zero lower bound, which limits the ability of central banks to stimulate the economy through standard interest rate policy, was reached in Japan in the 1990s and in the U.S. and Western Europe after the financial crisis in 2008. Buiter and Panigirtzoglou (1999, 2003), Goodfriend (2000), Mankiw (2009), Buiter (2009) and Menner (2011) have analyzed a tax on money holdings as a way of alleviating this problem. Importantly, the tax breaks the arbitrage condition in standard models that induces savers to hold cash instead of other financial assets when nominal interest rates go below zero, thus allowing for significantly negative nominal interest rates.

Perhaps less known is that a (periodic) tax on money holdings existed for almost 200 years in large parts of medieval Europe. Gesell taxes were implemented by coins being legal tender for only a limited time period and, at the end of the period, the coins had to be exchanged into new coins for an ex ante known fee - an institution known as renovatio monetae or periodic re-coinage; e.g., see Allen (2012, p.35).[1] In Gesell's original proposal, the holders of money had to buy and attach stamps to bank notes for them to retain their full nominal value. In the system with periodic re-coinage, the monetary authority ensured that the new coins could be distinguished from old coins by altering their physical appearance so that it would be easy to verify that only the new coins were legal tender.

There was substantial variation in the level of Gesell taxes. In Germany, four old coins were usually exchanged for three new coins, and the Gesell tax was 25 percent; in the Teutonic order, the tax was 17 percent, and in Denmark it was up to 33 percent; see Mehl (2011, p. 33), Paszkiewicz (2008) and Grinder-Hansen (2000, p. 85). Note also that, with periodic Gesell taxes, revenues depend not only on the fee charged at the time of the re-coinage but also on the duration of an issue. In specific currency areas, re-coinage could occur up to twice per year and involve annualized rates of up to 44 percent; see Kluge (2007).[2]

To generate revenues through seigniorage, the monetary authority benefits from creating an exchange monopoly for the currency. In a system with Gesell taxes and re-mintage,

[1] Also known as *coin renewals*.

[2] The annualized rate is based on a Gesell tax of 25 percent that was levied twice per year, as in, e.g., Magdeburg; see Mehl (2011, p. 33).

in addition to competing with foreign coin issuers, the monetary authority competes with its own older issues. To limit the circulation of illegal coins, the monetary authorities penalized the usage of invalid coins. Furthermore, fees, rents and fines had to be paid with current coins; see Haupt (1974, p. 29), Grinder-Hansen (2000, p. 69) and Hess (2004, p. 16–19). In addition to the system with Gesell taxes, there was also a system with long-lived coins in the High Middle Ages of Europe (1000–1300 A.D.), where the period when coins were legal was not fixed; see Kluge (2007, p. 62–64).[3]

The disciplines of archaeology and numismatics have long been familiar with periodic re-coinage (Kluge, 2007 , Allen, 2012, Bolton, 2012). Although scientific methods in archaeology and numismatics identify the presence of re-coinage, empirical evidence in written sources is scarce on the consequences of re-coinage with respect to prices and people's usage of new and old coins. However, evidence from coin hoards indicates that old (illegal) coins often but not always circulated with new coins; see Allen (2012, p. 520–23) and Haupt (1974, p. 29). In addition, written documents mention complaints against this monetary tax (Grinder-Hansen 2000, p. 51–52 and Hess, 2004, p. 19–20). Despite being common for an extended period of time, this type of monetary system has seldom if ever been analyzed theoretically in the economics or economic history literature.

The purpose of the present study is to fill this void in the literature. We formulate a cash-in-advance model in the spirit of Velde and Weber (2000) and Sargent and Smith (1997) to capture the implications of Gesell taxation in the form of periodic re-coinage on prices, returns and people's decisions to use new or old coins for transactions in an economy with varying degree of complexity. The model includes households, firms and a lord. Households care about consumption of goods and jewelry consumption. Households can trade goods either by bartering or by using money on the market. We capture complexity of the economy in terms of the number of goods that are traded. Bartering is costly in the sense that each bilateral meeting between traders carries a cost in terms of resources. When trading on the market, households face a cash-in-advance constraint. Households can hold both new and old coins, but only the new coins are legal tender. The firm can export goods in exchange for silver that is minted into coins, and coins can

[3] Sometimes, these coins were valid for the entire duration of the reign of the coin issuer. In these cases, successors occasionally minted variants of the same coin type. These variants are called immobilized types and could be valid for very long time periods - occasionally centuries - and survive through the reigns of several rulers.

be melted into silver that is exported to buy the consumption good. An issue of coins is only legal for a finite period of time. Old coins must be re-minted at the re-coinage date to be considered legal tender. The lord charges a fee when there is a re-coinage so that for each old coin handed in, the household receives only a fraction in return. Although illegal, old coins can be used for transactions. To deter the use of illegal coins, lord plaintiffs check whether legal means of payment are used in transactions. When old coins are discovered in a transaction by the lord plaintiffs, the coins are confiscated and re-minted into new coins. Thus, whether illegal coins circulate is endogenous in the model. The lord's revenues depend on the re-coinage (and mintage) fee, old coin confiscations and the duration of each coin issue. The lord uses the revenues to finance consumption expenditures.

Because re-coinage occurs at a given frequency and not necessarily in each time period, a steady state need not exist. Instead of analyzing steady states, we analyze a model where re-coinage occurs at fixed (and equal) time intervals. To focus on steady-state-like properties, we analyze cyclical equilibria, i.e., equilibria where the price level, money holdings, consumption, etc., are the same at a given point in different coin issues.

A key results is that the system with Gesell taxes works, in the sense that agents participate in re-minting coins and the system generates tax revenues, the less complex the economy is. The reason is that the share of goods being traded on the market is smaller, since it is easier to find a double coincidence of wants when bartering. This, in turn, increases the probability that illegal coins are detected when the economy is less complex and has a lower degree of monetization. Furthermore, the system with Gesell taxes also works 1) if the tax is sufficiently low, 2) if the time period between two instances of re-coinage is sufficiently long and 3) if the probability of being penalized for using old illegal coins is sufficiently high. Also, prices increase over time during an issue period and fall immediately after the re-coinage date. Moreover, the higher the Gesell tax is, the higher the price increases are (as long as the coins are surrendered for re-coinage). Additionally, although nominal returns become negative when the Gesell tax is levied, real returns are unchanged because the price level adjusts accordingly as a result of the reduction in money holdings.

The paper is organized as follows. In section 2, we provide some stylized facts regarding medieval European coins and discuss the concept of periodic re-coinage. The extension of short-lived coinage systems through time and space as well as seigniorage and enforcement of short-lived coinage systems are outlined in section 3. In section 4, we use a cash-in-advance model to analyze the consequences of periodic re-coinage. Finally, section 5 delineates the conclusions.

2 Short-lived coinage systems through time and space

2.1 The basics of medieval money

Money in medieval Europe was overwhelmingly in the form of commodity money, based on silver,[4] fiat money did not exist in its pure form. The control of the coinage, i.e., the right to mint, belonged to the droit de régale, i.e., the king/emperor. In addition to the right to determine, e.g., the design and the monetary standard, the coinage right encompassed the right to use the profits from minting and to decide which coins were legal tender; see Kluge (2007, p. 52). The right to mint for a region could be delegated, sold or pawned to other local authorities (local lords, laymen, churchmen, citizens) for a limited or unlimited time period; see Kluge (2007, p. 53). The size of each currency area was usually smaller than today and could vary substantially. All of England was a single currency area (after 975), whereas Sweden and Denmark each had 2–3 areas. In contrast, in France and Germany, there were many small currency areas.

2.2 The concept of periodic re-coinage

A commonly used monetary system in the middle ages was Gesell taxation in the form of periodic re-coinage. The main feature of such a re-coinage system is that coins circulate for a limited time, and at the end of the period, the coins must be returned to the monetary authority and re-minted for an ex ante known fee, i.e., a Gesell tax. Thus, coins can be "short-lived", in contrast to a "long-lived" monetary system in which the coins do not have a fixed period as a legal means of payment.

To obtain revenues from seigniorage, a coin issuer benefits from having an exchange monopoly in both long- and short-lived coinage systems. However, in a short-lived coinage

[4] The reason for this was the relative abundance of silver mines that lead to a high supply of silver; see Spufford (1988, p.109ff, 119ff).

system, the minting authority not only faces competition from other coin issuers but also from its own old issues that it minted. To create a monopoly position for its coins, legal tender laws stated that foreign coins were ipso facto invalid and had to be exchanged for the current local coins with the payment of an exchange fee in an amount determined by the coin issuer.[5] Moreover, only one local coin type was considered legal at a given point in time.[6] The frequency and exchange fee of re-coinage varied across regions (see section 3.2 below). To make it easy to verify current and invalid coins, the main design of the coin was changed, whereas the monetary standard largely remained unchanged. This is similar to Gesell's original proposal, where stamps had to be attached to a bank note for it to retain its full value, which made it easy to verify whether the tax had been paid.

Written documents about periodic re-coinage tell that coins were usually exchanged on recurrent dates at a substantial fee and that coins were only valid for a limited (and ex ante known) time. The withdrawals were systematic and recurrent. One may also want to distinguish between periodic re-coinage and coinage reform, which is a distinction that has not necessarily been made explicit by historians and numismatists.[7] When a coinage reform is undertaken, coin validity is not constrained by time. A coinage reform also includes a re-mintage but is announced infrequently, and the validity period of the coins is not (explicitly) known in advance. Moreover, the coin and the monetary standard are generally changed considerably.[8] Note that if the issuer charges a fee at the time of the reform, the coinage reform shares some features of re-coinage, but because the monetary standard is changed, there may be additional effects, e.g., on the price level at the time of the reform.

[5]In 1231, the German king Henry VII (1222–35) published an edict in Worms stating that in towns in Saxony with their own mints, goods could only be exchanged for coins from the local mint; see Mehl (2011, p. 33). However, when this edict was published, the system of coins constrained through time and space had been in force for a century in large parts of Germany.

[6]The coin issuer therefore has an incentive to ensure that foreign coins are not allowed to circulate. Moreover, to prevent illegal coins from circulating, the minting authority must control both the local market and the coinage; see Kluge (2007, p. 62–63).

[7]In fact, historians often use the term re-coinage for both periodic re-coinage and coinage reform.

[8]England had two re-mintings in the 13th century when the coinage was long-lived, but these events had other purposes than to simply charge a gross seigniorage. The short-cross pennies minted in the 12th and 13th centuries were often clipped. A re-minting occurred in 1247. A new penny was introduced ('long-cross') with the cross on the reverse extended to the edge of the coin to help safeguard the coins against clipping. Another coinage reform occurred in 1279. Before 1279, the double-lined cross on the long-cross pennies was used when cutting the coins into halves to obtain small change for the penny. New denominations were introduced in 1279 - all with single-lined crosses on the reverse. In addition to the new penny, groat, halfpence and farthing were also minted.

3 Seigniorage and enforcement of short-lived coinage systems

3.1 Geographic extension of short-lived coinage systems

There is a substantial historical and numismatic literature that describes the extent of periodic re-coinage; see, e.g., Kluge (2007), Allen (2012), Bolton (2012) and Svensson (2016). Three methods have been used to identify periodic re-coinage and its frequency, namely, written documents, the number of coin types per ruler and the years, and distribution of coin types in hoards (for details, see Svensson (2016), appendix). There is a reasonable consensus in determining the extension of long- and short-lived coinage systems through time and space. Long-lived coins were common in northern Italy, France and Christian Spain from 900–1300. This system spread to England when the sterling was introduced during the second half of the 12th century (see Map 1). In France, in the 11th and 12th centuries, long-lived coins dominated in most regions (the southern, western and central parts), and the rights to mint were distributed to many civil authorities. In northern Italy, where towns took over minting rights in the 12th century, long-lived coins likewise dominated; see Kluge (2007, p. 136ff)

Short-lived coinage systems were the dominant monetary system in central, northern and eastern Europe from 1000–1300. The first periodic re-coinage in Europe occurred in Normandie between 930 and 1100 (Moesgaard 2015). Otherwise a well-known example of periodic re-coinage is England. Compared to Normandie, the English short-lived coins were valid in a large currency area. Periodic re-coinage was introduced in the English kingdom in approximately 973 and lasted until around 1125; see Spufford (1988, p. 92) and Bolton (2012, p. 87ff).

The eastern parts of France and the western parts of Germany had periodic re-coinage in the 11th and 12th centuries; see Hess (2004, p. 19–20). However, the best examples of short-lived and geographically constrained coins can be found in central and eastern Germany and eastern Europe, where the currency areas were relatively small. Here, periodic re-coinage began in the middle of the 12th century and lasted until approximately 1300 and was especially frequent in areas where uni-faced bracteates were minted,[9] which

[9]Bracteates are thin, uni-faced coins that were struck with only one die. A piece of soft material, such

usually occurred annually but sometimes twice per year; see Kluge (2007, p. 63).

Sweden had periodic re-coinage of bracteates in two of three currency areas (especially in Svealand and to some extent in western Götaland) for more than a century, from 1180–1290. This conclusion is supported by evidence of numerous coin types per reign and the composition of coin hoards; see Svensson (2015). Denmark introduced periodic re-coinage in all currency areas in the middle of the 12th century, which continued for 200 years with some interruptions; see Grinder-Hansen (2000, p. 61ff). Poland and Bohemia had periodic re-coinage in the 12th and 13th centuries; see Sejbal (1997, p. 26), Suchodolski (2012) and Vorel (2000, p. 341).

3.2 Seigniorage and prices in systems with re-coinage

The seigniorage under re-coinage depends not only on the fee charged at the time of the re-coinage but also on the duration of an issue. Given a fee of, for example, 25 percent at each re-coinage, the shorter the duration is, the higher the revenues are, given that money holdings are not affected. Any reduction in money holdings because of a shortening of issue time would move revenues in the other direction.

There was a substantial variation in the level of seigniorage. In England from 973–1035, re-coinage occurred every sixth year. For approximately one century after 1035, English kings renewed their coinage every second or third year; see Spufford (1988, p. 92) and Bolton (2012, p. 99ff). The level of the fee is uncertain.[10]

In other areas in Europe, the duration was often significantly shorter. Austria had annual re-coinage until the end of the 14th century, and Brandenburg had annual re-coinage until 1369 (Kluge (2007, p. 108, 119)). Some individual German mints had bi-annual or annual renewals until the 14th or 15th centuries (e.g., Brunswick until 1412); see Kluge (2007, p. 105). In Denmark, re-coinage was frequent (mostly annual) from the middle of the 12th century and continued for 200 years with some interruptions; see Grinder-Hansen (2000, p. 61ff). Sweden had re-coinage beginning in approximately 1180 that continued for approximately one century; see Svensson (2015). In Poland, King

as leather or lead, was placed under the thin flan. Consequently, the design of the obverse can be seen as a mirror image on the reverse of the bracteates.

[10] According to Spufford (1988), four old coins were exchanged for three new coins, although this calculation is based on a rather uncertain weight analysis. If the gross seigniorage was 25 percent every sixth year, the annualized rate was almost 4 percent.

Table 1: Exchange fees and duration of re-coinage in different areas

Region	Currency area♦	Period	Gesell tax★ (Annualized)	Duration years★	Method/Source†
Normandie	Small	930–1000	n.a.	3–5	2–3,
	Small	ca. 1000–1100	n.a.	1–3	Moesgaard (2015)
England	Large	973–1035	n.a.	6	1–3, Spufford (1988)
	Large	1035–1125	n.a.	2–3	2–3, Bolton (2012)
Germany, western✠	Small	ca. 1000–ca. 1300	mostly 25% (4.6%–25%)‡	1–5	1–3, Hess (2004)
Germany, eastern, northern✠	Small	ca. 1140–ca. 1330, sometimes until 15th cent.	mostly 25% (25%–44%)‡	$\frac{1}{2}$ or 1	1–3, Kluge (2007)
Teutonic Order in Prussia	Medium	1237–1364	17% (1.6%)	10	1–3, Paszkiewicz (2008)
Austria	Small	ca. 1200–ca. 1400	n.a.	1	2–3, Kluge (2007)
Denmark	Medium	1140s–1330s.	33% (33%)	1, with interruptions	1–3, Grinder-Hansen (2000)
Sweden, Svealand	Large	1180–1290	n.a.	1–5	2–3, Svensson
Sweden, Götaland	Large	1180–1290	n.a.	3–7	(2013)
Poland	Small	ca. 1100–ca. 1150	n.a.	3–7	1–3,
	Small	ca. 1150–ca. 1200	n.a.	1	Suchodolski
	Medium	ca. 1200–ca. 1300	n.a.	$\frac{1}{3}$ or $\frac{1}{2}$	(2012)
Bohemia-Moravia	Medium	ca. 1150–1225	n.a.	1	Sejbal (1997) and
	Medium	1225–ca. 1300	n.a.	$\frac{1}{2}$	Vorel (2000)

Notes: ♦ We do not use a formal definition of area size. By a large area, we mean a country or a substantial part of a country, such as England or Svealand. A small area is usually a city and its hinterland. A medium-sized area is somewhere in between and is exemplified by the kingdom of Wessex. †Methods: 1) Written sources; 2) No. of types per time period; 3) Distribution of coin hoards. ✠ Various mints and authorities. ‡Annualized rate based on a fee of 25 percent. ★ When known.

Boleslaw (1102–38) began with irregular re-coinages - every third to seventh year, but later, these became far more frequent. At the end of the 12th century, coin renewals were annual, and in the 13th century, they occurred twice per year; see Suchodolski (2012). Bohemia also had re-coinage at least once each year in the 12th and 13th centuries; see Sejbal (1997, p. 83) and Vorel (2000, p. 26). In contrast, the Teutonic Order in Eastern Prussia had periodic re-coinages only every tenth year between 1237 and 1364; see Paszkiewicz (2008).

The exchange fee in Germany was generally four old coins for three new coins, i.e., a Gesell tax of 25 percent; see, e.g., Magdeburg (12 old for 9 new coins, Mehl, 2011 p. 85). In Denmark, the Gesell tax - three old coins for two new coins—was higher, at 33 percent; see Grinder-Hansen (2000, p. 179). The annualized tax in Germany could be very high

- up to 44 percent.[11] The Teutonic Order in Prussia had a relatively generous exchange fee of seven old coins for six new coins; see Paszkiewicz (2008). This fee represents a tax rate of almost 17 percent, or in annualized terms, 1.6 percent.

Unfortunately, evidence is scarce on the prices in monetary systems with re-coinage. Indeed, finding price indices for the period under discussion is almost impossible. However, some evidence from the Frankish empire indicates that prices rose during an issue.[12] Specifically, several attempts at price regulations that followed a re-coinage/coinage reform in 793–4 seem to indicate problems with rising prices; see Suchodolski (1983).

3.3 Success, monitoring and enforcement of re-coinage

There was considerable variation in the success of re-coinage. The coin hoards discovered to date can tell us a great deal about the success of re-coinage. In Germany, taxation was high and re-coinage occurred frequently; see table 1. Unsurprisingly, hoards in Germany from this period (1100–1300) usually contain many different issues of the local coinage as well as many issues of foreign coinage, i.e., locally invalid coins; see Svensson (2016) table 3. This indicates that the monetary authorities had problems enforcing the circulation of their coins. By missing some coin renewals and saving their retired coins, people could accumulate silver or use the old coins illegally. In contrast, hoard evidence from England indicates that the re-coinage systems were partly successful; see Dolley (1983). As shown in table 2, almost all of the coins in hoards are of the last type during the period from 973–1035, when coins were exchanged every sixth year. However, from 1035–1125, only slightly more than half of the coins were of the last type, which indicates that the system worked well up to 1035 but less so after that date. One reason for this result may be that the seigniorage for the later period was higher because of the shorter time period between withdrawals (at an unchanged exchange fee; see table 1).

Because hoards often contain illegal coins, the incentives to try to avoid re-coinage fees appear to occasionally have been rather high. To curb the circulation of illegal coins, monetary authorities used different methods to control the usage of coins. The usage of invalid coins was deemed illegal and was penalized, although the possession of

[11] The annualized rate is based on a Gesell tax of 25 percent levied twice per year, as in, e.g., Magdeburg; see Mehl (2011, p. 33).

[12] The Frankish empire seems to have had a system similar to re-coinage in the 8th and 9th centuries, although the weight of the coins was often changed when they were exchanged in this system.

Table 2: The composition of English coin hoards from 979–1125. Number of coin hoards, number of coins and shares

Period		973–1035		1035–1125	
Years between re-coinages		6 years		2–3 years	
		No. of coins	Share	No. of coins	Share
Coins from	Last issue	886	86.5%	8 771	54.3%
	Second to last issue	137	13.4%	1 724	10.7%
	Third to last issue	1	0.1%	698	4.3%
	Earlier issues	0	0.0%	4 964	30.7%
Total number of coins		1 024	100.0%	16 157	100.0%

Notes: Source Svensson (2016), table 2.

invalid coins was mostly legal.[13] If an inhabitant used foreign coins or old local coins for transactions and was detected, the penalty could be severe. Moreover, sheriffs and other administrators who accepted taxes or fees in invalid coins were penalized; see Haupt (1974, p. 29), Grinder-Hansen (2000, p. 69), and Hess (2004, p. 16). Controlling the usage of current coins was likely easier in cities than in the countryside.[14]

The minting authority could also indirectly control the coin circulation in an area. Documents show that fees, rents and fines were to be paid with current coins, in contrast to traditional situations where payment in kind was possible; see Grinder-Hansen (2000, p. 69), and Hess (2004, p. 19).

4 The model

In this section, we outline the model, define equilibria and analyze equilibrium outcomes in terms of how prices evolve. We also analyze under what conditions on re-mintage fees and issue length, old and new coins are used together.

[13] City laws in Germany stated that neither the mint master nor a judge was allowed to enter homes and search for invalid coins.

[14] As noted in sections 2.1 and 3.1, medieval currency areas could be large, such as in England and Sweden, or small, as in Germany and Poland. However, irrespective of the size of the currency area, systems with short-lived coins as legal tender could often be strictly enforced only in a limited area of the authority's domain, such as in cities. If most trade occurred in cities, this restriction may not be a strong constraint, however. Normally, the city border demarcated the area that included the jurisdiction of the city in the Middle Ages. The use of foreign and retired local coins within the city border was forbidden. This state of affairs is well documented in an 1188 letter from Emperor Friedrich I (1152–90) to the Bishop of Merseburg (Thuringia) regarding an extension of the city. The document plainly states that the market area boundary includes the entire city, not just the physical marketplaces; see Hess (2004, p. 16). A document from Erfurt (1248/51) shows that only current local coins could be used for transactions in the town, whereas retired local coins and foreign coins were allowed for transactions outside of the city border; see Hess (2004, p. 16).

4.1 The economic environment

The economy consists of households, firms and a lord. There are trade opportunities with the rest of the world and goods can be exchanged for silver on the world market at a fixed world market relative price γ. Households care about consumption of goods and jewelry consumption. Households can trade goods either by bartering or by using coins on the market. When trading on the market, households face a cash-in-advance constraint. Household money holdings consist of new and old coins, m_t^n and m_t^o, made of silver.[15] Only new coins are legal tender, but households can use both types in transactions. Thus, whether illegal (old) coins circulate is endogenous in the model. The new coins are withdrawn from circulation every T'th period. Specifically, to be considered legal tender after a withdrawal, coins must be handed in to be re-minted. Any coin that is not returned for re-mintage is not legal tender and is thus treated as an old coin after its withdrawal. Therefore, a given issue of coins is legal tender for T periods. The lord charges a Gesell tax τ at the time of each withdrawal. Specifically, for each coin handed in for re-mintage, each household receives $1-\tau$ new coins in return, and the lord gets the remainder. Although illegal, old coins can be used for transactions, but because of the possibility of punishment for using illegal coins, it is costly to do so. We model the punishment for using illegal coins as follows. There are lord plaintiffs that check whether the legal means of payment are used in transactions. If old coins are discovered in a transaction by the lord plaintiffs, they are confiscated by the lord plaintiffs, re-minted as new coins and used to fund the lord's expenditures. We let e_t denote the exchange rate between old and new coins. The probability of avoiding detection is increasing in aggregate real money holdings, i.e., in the total amount households purchase in the market. Specifically, the avoidance probability is $\chi\left(\frac{m_t}{p_t}\right)$ where $\frac{m_t}{p_t}$ is the real value of aggregate household money holdings ($m_t = m_t^n + e_t m_t^o$). The lord plaintiffs find old coins with probability $1-\chi\left(\frac{m_t}{p_t}\right)$. Because of the confiscation of old coins by the lord plaintiffs, old and new coins need not circulate at par. The firm can melt (mint) coins and export (import) silver in exchange for the consumption goods. The lord's revenues, i.e., from minting, re-mintage and confiscations, are spent on the lord's consumption, denoted as g_t.

Along the lines of Velde and Weber (2000) and Sargent and Smith (1997), competitive

[15] For simplicity, we ignore foreign coins.

firms can produce: 1) a consumption good c_t using the endowment or by exporting silver; 2) jewelry d_t from melting coins or importing silver; and 3) new coins by importing silver or melting old coins.[16] At the beginning of a period t, households has an endowment of goods ξ_t, a stock of jewelry d_t, and a stock of new and old coins respectively. A share of the endowment of the household is sold to the firms in return for a claim on firm profits. The rest of the endowment is used in bartering, described in detail below. Then, shopping (and bartering) begins with households using coin balances to buy consumption and jewelry at competitively determined prices p_t and q_t, respectively. Firms sell the goods endowment to households and the lord, and receive coins in exchange. Moreover, n_t^n coins are minted for the households and μ_t^n new coins and μ_t^o old coins are melted. If coins are minted, firms pay the same fee as when coins are returned on the re-coinage date. Then, the profits are returned to the households in the form of dividends. Finally, on the re-coinage date, households decide on the number of coins h_t^r that is to be handed in to the firm for re-minting into new coins.

4.1.1 The firm

The firm profits are

$$\Pi_t = p_t \left(c_t^M + \mathrm{Im}_t + g_t \right) + (1-\tau) \left(n_t^n + q_t^r n_t^r \right) - \mu_t^n - e_t \mu_t^o + q_t h_t - h_t^r, \tag{1}$$

where c_t^M denotes household consumption goods bought on the market, Im_t is net silver exports, g_t is lord consumption in period t, n_t^n is minting of new (household) coins, n_t^r recoined coins, q_t^r the mint price of recoined coins, μ_t^n and μ_t^o denote melting of new and old (household) coins, new coins from household re-coinage with h_t^r being the amount handed in for re-coinage and $1/q_t^r$ the corresponding mint price and

$$h_t = b \left(\mu_t^n + \mu_t^o - n_t^n \right) - \gamma \, \mathrm{Im}_t \tag{2}$$

where γ is the relative world market price of silver. Mintage must be non-negative and melting cannot exceed the stock of new and old coins m_t^n and m_t^o, respectively. Moreover,

[16] A motivation for competitive mints is that, e.g., in the 11th–12th centuries, England had at up to approximately 70 active mints active at some points; see Allen (2012, p. 16 and p. 42f). Moreover, these mints were sometimes farmed out; see Allen (2012, p. 9).

coins are defined by the number b of grams of silver per coin. Thus, the firm faces the following constraints, related to mintage and melting, $n_t^n \geq 0$, $\mu_t^n \geq 0$ and $\mu_t^o \geq 0$. The firm maximizes its profits in (1) subject to these constraints and

$$c_t^M + g_t \leq \xi_t + \text{Im}_t \,. \tag{3}$$

From the firm's first-order condition for minting, using (2), if $\frac{1-\tau}{b} > q_t$ then $n_t^n = \infty$, if $\frac{1-\tau}{b} < q_t$ then $n_t^n = 0$ and if

$$\frac{1-\tau}{b} = q_t \text{ then } n_t^n \in [0, \infty). \tag{4}$$

Equilibrium then requires that $\frac{1-\tau}{b} \leq q_t$ with equality, whenever $n_t^n > 0$.

Firm optimization leads to the following conditions for the melting of new coins; if $\frac{1}{b} < q_t$ then $\mu_t^n = \infty$, if $\frac{1}{b} > q_t$ then $\mu_t^n = 0$ and if

$$\frac{1}{b} = q_t \text{ then } \mu_t^n \in [0, \infty). \tag{5}$$

Repeating the same for μ_t^o gives, if $\frac{e_t}{b} < q_t$ then $\mu_t^o = \infty$, if $\frac{e_t}{b} > q_t$ then $\mu_t^o = 0$ and if

$$\frac{e_t}{b} = q_t \text{ then } \mu_t^o \in [0, \infty). \tag{6}$$

Firm optimization regarding imports implies, if $p_t > q_t\gamma$ then $Im_t = \infty$, if $p_t < q_t\gamma$ then $Im_t = -\infty$ and if

$$p_t = q_t\gamma \text{ then } Im_t \in (-\infty, \infty). \tag{7}$$

Finally, noting that $n_t^r = h_t^r$, the first-order condition regarding re-coinage is, if $q_t^r < \frac{1}{1-\tau}$ then $n_t^r = \infty$, if $q_t^r > \frac{1}{1-\tau}$ then $n_t^r = 0$ and if

$$q_t^r = \frac{1}{1-\tau} \text{ then } n_t^r \in [0, \infty). \tag{8}$$

4.1.2 The household

Although we have described consumption as a single aggregate consumption good above, akin to Khan, King, and Wolman (2003), we now reinterpret the aggregate good as involving a finite number K of individual products. As to standard modern macro models, see e.g., Erceg, Henderson, and Levin (2000) there are several households. Each individual household is endowed with an amount ξ of particular good k, with K different types. The share of households endowed with good k is $\frac{1}{K}$.

The household keeps a share $1-\phi_t$ of the endowment for use in barter and, following Velde and Weber (2000) and Sargent and Smith (1997), hands in remainder to the firm in return for a share in firm profits. As we will see below, ϕ_t will capture the degree of monetization in the economy. As in Khan, King, and Wolman (2003), the firm repackages the good into an aggregate good (that consists of all types of the K goods) that is purchased by consumers. The share handed in to the firm is traded on a competitive market and the rest is used in barter. Several members of a household takes part in bartering on behalf of the household. A household member (we think of households consisting of several members) is assigned to trade a particular good $j \neq k$ and brings a fraction of $(1-\phi_t)\,\xi$ of the endowment in return. For simplicity, we assume that the barter trades only of the direct barter type as described in Oh (1989). In each time period, there is an infinite number of bartering rounds. Each household member is randomly selected to meet a member of another household. The cost when bartering depends on e.g., transportation costs, and is increasing with the amount of good used in bartering. Specifically, the cost for N_r meetings is given by $\hat{w}\left(\left(1-\phi_t\right)N_r\right)$.[17] The probability that the member meets a member of another household with the desired good is $p=\frac{1}{K}$. The probability of double coincidence of wants is then $\frac{1}{K^2}$. Let N_r denote the number of meetings until a successful barter is made and let $f_N\left(N_r,K\right)$ denote the probability that an agent has a double coincidence of wants after exactly N_r rounds. Then $f_N\left(N_r,K\right)=\left(1-p^2\right)^{N_r-1}p^2$ where $p=\frac{1}{K}$. The expected utility cost of barter for the household is then

$$\int_0^1 \sum_{N_r=1}^{\infty} \hat{w}\left(\left(1-\phi_t\right)N_r\right) f_N\left(N_r,K\right) di \tag{9}$$

where $\hat{w}$ satisfies $\lim_{N_r\to\infty}\hat{w}\left(\left(1-\phi_t\right)\bar{l}N_r\right)f_N\left(N_r\right)=0$ and the integral is over the household members. Define

$$w\left(\left(1-\phi_t\right),K\right)=\int_0^1 \sum_{N_r=1}^{\infty} \hat{w}\left(\left(1-\phi_t\right)N_r\right)\left(1-\frac{1}{K^2}\right)^{N_r-1}\frac{1}{K^2}di. \tag{10}$$

[17] Note that market transaction might be costly for similar reasons, although less so, since fewer meetings are required. For simplicity, we normalize the cost of going to the market to zero.

The following Lemma shows that w is increasing in K. Thus, the cost of barter increases in the number of goods. The reason is that the expected number of meetings for a double coincidence of wants increases.

Lemma 1 *w is increasing in K and w_1 is decreasing in K.*

Proof: See the appendix. ■

Also, since $\hat{w}$ is strictly concave, w is concave in it's first argument. In the following, we assume that w is strictly concave.

The household preferences are

$$\sum_{t=0}^{\infty} \beta^t \left[u(c_t) + w((1-\phi_t), K) + v(d_{t+1}) \right]. \tag{11}$$

Both u and v are assumed to be strictly increasing and strictly concave. We impose the standard Inada conditions so that $\lim_{c\to 0} u'(c) \to \infty$, $\lim_{x\to 0} \hat{w}(x) \to \infty$ and $\lim_{d\to 0} v'(d) \to \infty$. Households own an endowment ξ_t of the consumption good. Following Velde and Weber (2000) and Sargent and Smith (1997), the endowment is transferred to firms in return for a claim on profits. The household maximizes utility in (11), subject to the CIA constraint

$$p_t \phi_t c_t + q_t h_t = m_t^n + e_t \chi \left(\frac{m_t}{p_t} \right) m_t^o, \tag{12}$$

the budget constraint

$$\begin{aligned} ((1-\mathbb{I}_t) + \mathbb{I}_t q_t^r) m_{t+1}^n + e_t m_{t+1}^o \leq & \; (1-\mathbb{I}_t)\Pi_t^n + \mathbb{I}_t h_t^{rh} + e_t \left(\Pi_t^o + \mathbb{I}_t \left(\Pi_t^n - h_t^{rh} \right) \right) \\ & + m_t^n + e_t \chi \left(\frac{m_t}{p_t} \right) m_t^o - p_t \phi_t c_t - q_t h_t. \end{aligned} \tag{13}$$

where $\mathbb{I}_t = 1$ if $t = T, 2T, 3T$ and 0 otherwise, Π_t^n are firm dividends in new coins and Π_t^o dividends in old coins. Note that $h_t^{rh} \in [0, \Pi_t^n]$. Also, $c_t \geq 0$, $m_{t+1}^n \geq 0$ and $m_{t+1}^o \geq 0$. Furthermore, $h_t^{rh} \in [0, m_t^n]$ if $t = T$ and $h_t^{rh} = 0$ otherwise.

We now derive the household Euler equation. Using the first-order condition with respect to c_t and h_t, the first-order condition with respect to d_{t+1} can be written as[18]

$$\frac{u'(c_t)}{\phi_t} \frac{q_t}{p_t} = \beta \frac{u'(c_{t+1})}{\phi_{t+1}} \frac{q_{t+1}}{p_{t+1}} + v'(d_{t+1}). \tag{14}$$

[18] Note that since jewelry is a consumer durable good, the Euler equation here is similar to Euler equations in such models; see e.g., equation (5) in Barsky, House, and Kimball (2007).

As usual, the Euler equation describes the consumption-savings trade-off in the model. To get the intuition behind expression (14), consider a consumer that chooses to save some more by reducing consumption today and holding some extra jewelry, in order to increase consumption tomorrow. The decrease in consumption today leads to a decrease in utility through $u'(c_t)$, and is transformed into jewelry at the relative price $\frac{q_t}{p_t}$. When holding some extra jewelry, this gives the consumer a direct payoff effect through $v'(d_{t+1})$ and an indirect effect through an increase in consumption tomorrow. The change if $u'(c_{t+1})$ is discounted by β and the stored jewelry is sold at the relative price $\frac{q_{t+1}}{p_{t+1}}$.

Here, we describe the household optimality conditions, assuming $c_t > 0$ and $p_t > 0$ for all t, which holds in equilibrium.[19] Whether old or new coins are held depend on how exchange rates affect their relative return. Using the first-order conditions with respect to c_t and m^n_{t+1}, if $m^o_{t+1} > 0$ then

$$((1-\mathbb{I}_t) + \mathbb{I}_t q^r_t)\, e_{t+1}\chi\left(\frac{m_{t+1}}{p_{t+1}}\right) \geq e_t. \tag{15}$$

Since the consumer holds old coins in period $t+1$, the exchange rates in periods t and $t+1$ have to give the consumer incentives not to only hold new coins. Then, it follows that the exchange rate has to increase by at least $1/\chi\left(\frac{m_{t+1}}{p_{t+1}}\right)$ between adjacent periods, except in the withdrawal period when it appreciates by $1/q^r_t\chi\left(\frac{m_{t+1}}{p_{t+1}}\right)$. The appreciation of the exchange rates compensates the consumer for the loss due to confiscations by the lord plaintiff so that the consumer does not lose in value terms by holding an old coin, relative to new coins, for an additional period. The condition is slightly different for the withdrawal period, due to the fact that the return on holding new coins changes due to the tax on coins handed in for re-mintage.

If $m^n_{t+1} > 0$, if $t \neq T+1, 2T+1$ etc.,

$$((1-\mathbb{I}_t) + \mathbb{I}_t q^r_t)\, e_{t+1}\chi\left(\frac{m_{t+1}}{p_{t+1}}\right) \leq e_t. \tag{16}$$

Since the consumer now holds new coins in period $t+1$, the exchange rates in period t and $t+1$ have to give the consumer incentives to not only hold old coins. For this to be the case, the exchange rate increase cannot be too large and is bounded above by

[19] Some additional first-order conditions are illustrated in Appendix A.1.

$1/\left(\left(1-\mathbb{I}_t\right)+\mathbb{I}_t q_t^r\right)\chi\left(\frac{m_t}{p_t}\right)$.

Finally, the household also optimally chooses the share of coins to be handed in for re-coinage, h_t^{rh} in periods $t \neq T, 2T$ etc; if $e_t < 1$ then $h_t^{rh} = \infty$, if $e_t > 1$ then $h_t^{rh} = 0$ and if

$$e_t = 1 \text{ then } h_t^{rh} \in [0, \infty). \tag{17}$$

When choosing how to allocate the new coins in period T to new and old coins in the next period, the household takes into account its relative value. When handing in a coin for re-mintage, the value is one while when not handing it in, the value is e_t. Thus, if $e_t < 1$ is low enough, all new coins are re-minted and if $e_t > 1$, no new coins are re-minted.

The first-order conditions with respect to ϕ_t is

$$\frac{u'\left(c_t\right)}{\phi_t} c_t + w_1\left(\left(1-\phi_t\right), K\right) = 0. \tag{18}$$

Thus, the consumer chooses ϕ_t so that the cost of tightening the budget constraint (through an increase in trading on the market) is equal to the reduction in barter costs.

4.1.3 The lord

The lord gets revenue from coin withdrawals and confiscation of illegal coins. The lord hands in all confiscated old coins to the firms for them to be minted into new ones. Letting $m_t^L \geq 0$ denote coins stored by the lord, the lord budget constraint is

$$m_{t+1}^L = \tau\left(n_t^n + h_t^{rL} + \mathbb{I}_t h_t^{rh}\right) + \frac{1}{q_t^r} h_t^{rL} + \left(1-\mathbb{I}_t\right) m_t^L - p_t g_t. \tag{19}$$

where

$$h_t^{rL} = \left(1 - \chi\left(\frac{m_t}{p_t}\right)\right) m_t^o + \mathbb{I}_t m_t^L \tag{20}$$

Thus, the lord uses revenues from money withdrawals through h_t^{rh}, from new mintage through n_t^n, confiscations through m_t^o in (20) and previously stored coins m_t^L to spend on consumption g_t and coins stored to the next period m_{t+1}^L. In equilibrium, government spending is determined by the revenues generated by the Gesell tax τ and the plaintiff confiscation probability $1 - \chi\left(\frac{m_t}{p_t}\right)$. Since we restrict attention to steady state-like equilibria, see definition 2 below, we restrict g_t to be constant over time.

4.1.4 Money transition and resource constraints

Underlying the money transition equations are firm and household decisions as described above. When trading goods and jewelry, households spend $m_t^n + e_t\chi\left(\frac{m_t}{p_t}\right)m_t^o - q_t h_t$ on goods, which is equal to firm profits. After trading, households get dividends from the firms. Hence, new coin dividends are $m_t^n + p_t g_t - q_t b\left(\mu_t^n - n_t^n\right)$ and old coin dividends $e_t\chi\left(\frac{m_t}{p_t}\right)m_t^o - q_t b\mu_t^o$. Hence, the household stocks of new and old coins evolve according to, using that h_t^{rh} coins handed in for re-coinage gives $\frac{1}{q_t^r}h_t^{rh} = (1-\tau)\,h_t^{rh}$ new coins in return,

$$
\begin{aligned}
m_{t+1}^n &= (1-\mathbb{I}_t)\left(m_t^n + p_t g_t + (1-\tau)\,n_t^n - \mu_t^n\right) + \mathbb{I}_t\,(1-\tau)\,h_t^{rh} && (21)\\
m_{t+1}^o &= \chi\left(\frac{m_t}{p_t}\right)m_t^o - \mu_t^o + \mathbb{I}_t\left(m_t^n - h_t^{rh}\right). && (22)
\end{aligned}
$$

We also have the re-coinage constraint $h_t^r = h_t^{rh} + h_t^{rL}$.

Finally, we have the goods resource constraint

$$c_t + g_t = \xi_t + \mathrm{Im}_t \tag{23}$$

and the silver resource constraint

$$b\left(m_t^n + m_t^L\right) + d_t = S_t - \gamma\,\mathrm{Im}_t\,. \tag{24}$$

4.2 Equilibria

Definition 1 *An equilibrium is a collection* $\{m_{t+1}^n\}$, $\{m_{t+1}^o\}$, $\{m_{t+1}^L\}$, $\{n_t^n\}$, $\{\mu_t^n\}$, $\{\mu_t^o\}$, $\{n_t^L\}$, $\{\mu_t^L\}$, $\{\phi_t\}$, $\{c_t\}$, $\{g_t\}$, $\{d_{t+1}\}$, $\{\mathrm{Im}_t\}$, $\{h_t\}$, $\{h_t^{rh}\}$, $\{h_t^{rL}\}$, $\{p_t\}$, $\{q_t\}$ *and* $\{e_t\}$ *such that i) the household maximizes (11) subject to (12), (13),* $h_t^{rh} \in [0, \Pi_t^n]$*, the boundary constraints and the jewelry constraint; ii) the firm maximizes (1) subject to it's boundary constraints and (3); iii)* $c_t + g_t = \xi_t + \mathrm{Im}_t$ *and that (21), (22), (19) and (24) hold.*

For the rest of the analysis, we assume that the endowment is constant; $\xi_t = \xi$. Also, $S_t = S$ for all t and hence, using (20), the jewelry stock evolves according to

$$d_{t+1} = d_t + h_t. \tag{25}$$

For the lord, the budget is balanced over the cycle and $m_t^L < S_t - \gamma \operatorname{Im}_t$ for all t. Thus, summing the lord constraint (19) over $t = 1$ to T

$$\sum_{t=1}^{T} p_t g_t = \tau h_T^{rh} + \tau \sum_{t=1}^{T} n_t^n + \sum_{t=1}^{T} \left(1 - \chi\left(\frac{m_t}{p_t}\right)\right) m_t^o. \tag{26}$$

Note that due to the fact that money withdrawals occur infrequently, i.e., every T'th period, a steady state cannot be expected to exist. Therefore, we instead restrict the attention to *cyclical equilibria*. Thus, consider an issue with length T where an issue starts just after a withdrawal and ends just before the next withdrawal. Let $L_r^T = \{\tilde{r} : \tilde{r} = nT + r$ for $n \in N^+\}$ denote all time periods corresponding to a given period r in some issue.

Definition 2 *Given that money withdrawals occur every T'th period, an equilibrium is said to be* ***cyclical*** *if it satisfies* $m_{\hat{r}}^n = m_{\bar{r}}^n$, $m_{\hat{r}}^o = m_{\bar{r}}^o$, $m_{\hat{r}}^L = m_{\bar{r}}^L$, $n_{\hat{r}}^n = n_{\bar{r}}^n$, $\mu_{\hat{r}}^n = \mu_{\bar{r}}^n$, $\mu_{\hat{r}}^o = \mu_{\bar{r}}^o$, $c_{\hat{r}} = c_{\bar{r}}$, $\phi_{\hat{r}} = \phi_{\bar{r}}$, $d_{\hat{r}} = d_{\bar{r}}$, $\operatorname{Im}_{\hat{r}} = \operatorname{Im}_{\bar{r}}$, $h_{\hat{r}} = h_{\bar{r}}$, $h_{\hat{r}}^{rh} = h_{\bar{r}}^{rh}$, $h_{\hat{r}}^{rL} = h_{\bar{r}}^{rL}$, $p_{\hat{r}} = p_{\bar{r}}$, $q_{\hat{r}} = q_{\bar{r}}$ *and* $e_{\hat{r}} = e_{\bar{r}}$ *for all* $r \in \{1, \ldots, T\}$ *such that* $\hat{r}, \bar{r} \in L_r^T$.

The definition of cyclicality requires that, at the same point in two different issues and, the variables attain the same value, i.e., e.g., $m_{\hat{r}}^n = m_{\bar{r}}^n$.

We now proceed to analyze properties of equilibria. The following Lemma states that imports are zero in a cyclical equilibrium.

Lemma 2 *Imports are zero,* $\operatorname{Im}_t = 0$ *for all* t.

Since imports are zero and government spending is constant, consumption is also constant for all t. Moreover, from (18), ϕ_t is the same for all t.

Corollary 1 *Consumption and the share of consumption goods bought on the market,* ϕ_t, *is constant over the cycle,* $c_t = \xi - g$ *for all* t.

The below example illustrates how to find a cyclical equilibrium when there is a withdrawal of coins every second period.

Example 1 *Withdrawals occur every second period and only new coins are held in equilibrium. Also, for simplicity, we set* $m_1^L = 0$. *We first show that minting is zero in equilibrium. Noting that if* $n_1^n > 0$ *then, by cyclicality, we have* $\mu_2^n = n_1^n > 0$, *and hence,* *using (4) and (5),* $q_1 = \frac{1-\tau}{b}$ *(from competition between firms) and* $q_2 = \frac{1}{b}$*. Thus, using the CIA constraint (12) and the money transition equation (21) we have, using cyclicality (i.e.,* $m_3^n = m_1^n$*),*

$$\begin{aligned} p_1(\phi_1 c_1 + g) &= m_2^n \\ p_2(\phi_2 c_2 + g) &= \frac{1}{1-\tau} m_1^n \end{aligned} \tag{27}$$

for $t = \{1, 2\}$*. A similar result can be established when* $n_2^n > 0$ *and when* $n_1^n = n_2^n = 0$*. There are three candidate equilibria; i)* $n_1^n > 0$*,* $n_2^n = 0$ *and* $\mu_1^n = 0$*,* $\mu_2^n = n_1^n$*; ii)* $n_2^n > 0$*,* $n_1^n = 0$ *and* $\mu_1^n = n_2^n$*,* $\mu_2^n = 0$*; iii)* $n_t^n = \mu_t^n = 0$ *for* $t = 1, 2$*.*
First, suppose that $n_1^n > 0$ *so that* $q_1 = \frac{1-\tau}{b}$ *and* $q_2 = \frac{1}{b}$ *and thus* $p_1 = \gamma \frac{1-\tau}{b}$ *and* $p_2 = \gamma \frac{1}{b}$*. Then, since* $\mathrm{Im}_1 = \mathrm{Im}_2 = 0$ *we have* $c_1 = c_2$ *and* $\phi_1 = \phi_2$ *and thus, using (27),* $m_1^n = m_2^n$*, contradicting* $m_2^n = m_1^n + p_1 g + (1-\tau) n_1^n$*. Second, suppose that* $n_2^n > 0$ *so that* $q_2 = \frac{1-\tau}{b}$ *and* $q_1 = \frac{1}{b}$ *and thus* $p_2 = \gamma \frac{1-\tau}{b}$ *and* $p_1 = \gamma \frac{1}{b}$*. Then again using (27),* $m_2^n = \frac{1}{(1-\tau)^2} m_1^n = \frac{1}{(1-\tau)^2}(1-\tau)(m_2^n + p_2 g + (1-\tau) n_2^n)$*, a contradiction.*

The reason why an equilibrium does not exist is that the positive minting in period 1 implies that the return on money between period 1 and 2 is low, implying that n_1^n should be zero. The equilibrium where $n_2^n > 0$ can also be ruled out. Thus, the only equilibrium has $n_t^n = \mu_t^n = 0$ *for* $t = 1, 2$. Since the equilibrium entails neither minting nor melting, using money transition (21) $m_1^n = m_2^n + p_2 g$, we get that $m_1^n > m_2^n$, in turn implying that prices increase over the cycle (i.e., $p_2 > p_1$) following from a modified quantity theory argument using expression (27).[20]

The result in expression (27) can be shown to hold generally. By using money transition (21) in the CIA constraint (12), we can derive the following Lemma, akin to expression (27) in example 1.

Lemma 3 *The CIA constraint (12) is, when* $t \neq T$

$$p_t(\phi_t c_t + g) = m_{t+1}^n + e_t m_{t+1}^o \tag{28}$$

[20]Instead of the usual (12).

and, when $t = T$ *and* $h_t^{rh} > 0$

$$p_t\left(\phi_t c_t + g\right) = \frac{1}{1-\tau} m_{t+1}^n + e_t m_{t+1}^o \tag{29}$$

and

$$p_t\left(\phi_t c_t + g\right) = \frac{1}{1-\tau} e_t m_{t+1}^n + \left(1 - e_t\right)\left(m_t^n + p_t g + \left(1-\tau\right) n_t^n - \mu_t^n\right) + e_t m_{t+1}^o \tag{30}$$

otherwise.

Proof: See the appendix. ■

We now show that there is neither minting nor melting in equilibrium.

Lemma 4 *There is no mintage of new coins.*

To see this, suppose that only new coins are held so that $m_t^o = \mu_t^o = 0$ for all t. It is convenient to rearrange the Euler equation (14) as, using that consumption is constant,

$$p_t = Q_t\left(q_t, q_{t-1}, d_t, p_{t-1}\right) p_{t-1} \tag{31}$$

where

$$Q_t\left(q_t, q_{t-1}, d_t, p_{t-1}\right) = \beta \frac{q_t u'(c)}{q_{t-1} u'(c) - v'(d_t)\, p_{t-1}}. \tag{32}$$

Now, let us look at why the mintage must be zero. If $n_t^n > 0$ for some t then, using money transition (21) and Lemma 3, we have $m_{t+1}^n > \frac{\phi(\xi - g) + g}{\phi(\xi - g)} m_t^n$ and then, by Lemma 3, prices increase so that $Q_t > 1$. Since $n_t^n > 0$ we have $q_t = \frac{1-\tau}{b}$ and $q_{t+1} \geq \frac{1-\tau}{b}$ and thus, using that $p_t > p_{t-1}$, from (32), we have $Q_{t+1} > Q_t$. Then, prices in the next period increase even more. Money transition (21) and Lemma 3 then imply that there is positive mintage also in the next period. For the final period, a slightly different argument has to be used; see the proof for details. Induction then establishes that mintage is positive in all periods, thus violating cyclicality. We have the following corollary.

Corollary 2 *There is no melting of either new or old coins. Also, using the CIA constraint,* $\chi\left(\frac{m_t}{p_t}\right) = \chi\left(\phi\left(\xi - g\right)\right)$ *is constant for all* t.

Note that, from Lemma 2, Lemma 4 and the Corollary above, imports are zero and there are no jewelry transactions in equilibrium. Hence, total market trade is $\phi\xi$ and thus ϕ captures the degree of monetization in the economy.

Example 1 continued. *We now describe equilibrium prices. From cyclicality, money transition (21), that* $\mathrm{Im}_t = 0$ implies $c_t = \xi - g$, $\phi_1 = \phi_2$ *and the CIA constraint (12) that* $m_2^n = \frac{\phi(\xi-g)+g}{\phi(\xi-g)} m_1^n$. *Also, let* $\bar{\chi} = \chi\left(\frac{m_1}{p_1}\right) = \chi\left(\frac{m_2}{p_2}\right) = \chi\left(\phi\left(\xi - g\right)\right)$ where ϕ is determined by (18). *Moreover, using money transition (21) and (27), we have* $m_1^n = (1-\tau)\frac{\phi(\xi-g)+g}{\phi(\xi-g)} m_2^n$ and hence $\frac{\phi(\xi-g)}{\phi(\xi-g)+g} = \sqrt{1-\tau}$. *Then, goods prices increase by* $\frac{1}{\sqrt{1-\tau}}$ *between periods* 1 *and* 2*;*

$$p_2 = \frac{1}{\sqrt{1-\tau}} p_1. \tag{33}$$

Since $q_2 \leq \frac{1}{b}$*, any combination of jewelry prices such that* $q_2 = \frac{1}{\sqrt{1-\tau}} p_1$ *where* $q_1 \in \left[\frac{1-\tau}{b}, \frac{\sqrt{1-\tau}}{b}\right]$ *is feasible. Each such jewelry price is associated with a unique level of money holdings via the Euler equation. Finally, consider exchange rate restrictions for the equilibrium. Since households hold only new coins, from cyclicality, using* $q_t^r \geq \frac{1}{1-\tau}$*, (16), (16) and the household optimality condition for* h_2^{rh}*, we have* $e_1\bar{\chi} \leq (1-\tau)\, e_2$, $e_2\bar{\chi} \leq e_1$ and $e_2 \leq 1$*. Combining gives the following requirement for households to hold only new coins in equilibrium;*

$$1 - \tau \geq \bar{\chi}^2. \tag{34}$$

In general, the growth rate of prices can be easily computed from the CIA constraint and Lemma 3. From the CIA constraint we have

$$p_t \phi c = m_t^n + e_t \bar{\chi} m_t^o \tag{35}$$

and, from Lemma 3

$$p_{t-1}\left(\phi c + g\right) = m_t^n + e_{t-1}\bar{\chi} m_{t-1}^o \tag{36}$$

and hence, using that if $m_t^o > 0$ we have $e_t\bar{\chi} = e_{t-1}$ and $m_t^o = \bar{\chi} m_{t-1}^o$,

$$\frac{p_t}{p_{t-1}} = \frac{\phi c + g}{\phi c}. \tag{37}$$

We have the following theorem.

Theorem 3 *A cyclical equilibrium exists and entails* $n_t^n = \mu_t^n = \mu_t^n = 0$ *for all* t*. From*

Corollary 2, $\chi\left(\frac{m_t}{p_t}\right) = \chi\left(\phi\left(\xi - g\right)\right) \equiv \bar{\chi}$*. If* $1-\tau > \bar{\chi}^T$ *(*$1-\tau < \bar{\chi}^T$*), in any cyclical equilibrium, only new (both new and old) coins are held. If* $1-\tau = \bar{\chi}^T$ *either only new or both new and old coins are held. In any equilibrium, prices increase during an issue, i.e.,* $p_t > p_{t-1}$ *for* $t = 2, \ldots, T$ *and drop between periods* T *and* $T+1$*. If* $1-\tau \geq \bar{\chi}^T$ *prices increase at the rate* $(1-\tau)^{-\frac{1}{T}}$ *during a cycle and if* $1-\tau < \bar{\chi}^T$ *prices increases be the rate in (37) (at most at the rate* $\frac{1}{\bar{\chi}}$*) and no coins are handed in for re-coinage.*

Proof: See the appendix. ■

Suppose that only new coins are held. The results for increasing prices follow from the fact that money transition (21) implies that household money holdings increase over the cycle, due to the fact that firm dividends from government consumption increase household money holdings, so that, using a quantity theory argument and Lemma 3, prices increase. A modification of this argument establishes a similar result when also old coins are held. As long as only new coins are held, price increases are higher the higher is the Gesell tax, since a higher Gesell tax leads to higher government spending and, in turn, a higher increase in household money holdings during a cycle. When $1-\tau < \bar{\chi}^T$ so that old coins are also held, price increases depend on the plaintiff confiscation rate χ. The reason is that since no coins are handed in for re-coinage, the only source of government revenues is the confiscation of illegal coins and thus, $\bar{\chi}$ determines government spending and hence, of the increase in money holdings during a cycle.[21]

Since the nominal return is $\frac{p_1}{p_T} = 1-\tau$ when the Gesell tax is levied, nominal returns can be substantially negative - empirical evidence on the tax indicate that the implied yearly returns is as low as -44 percent at the date of tax collection. However, since goods prices fall simultaneously, due to the reduction in money holdings, real returns are unchanged.

The cutoff values for whether old coins are held depend on χ and τ. The reason for these cutoffs is that, assuming that both types are held, using (15) and (16), the exchange rate must appreciate at rate $1/\bar{\chi}$ when there is no re-coinage and at rate $\frac{1}{\bar{\chi}q_t^T}$ at

[21] Note that the value of old coins is indeterminate in equilibrium; see the proof for details. Hence the price level is also indeterminate, as it depends on the exchange rate; see (12). This in turn implies that government spending depends on the exchange rate and that spending is the highest when the exchange rate is at it's lowest possible level, i.e., $e_T = 1$. If this is the case, prices grow by $\bar{\chi}$ and otherwise the growth rate is lower, as the increase in private sector money holdings over the cycle is lower; see (21).

the re-coinage date. Using (17) when h_T^{rh} is interior we have $e_T = 1$ and hence

$$e_1 = \bar{\chi} e_2 = \cdots = \bar{\chi}^T. \tag{38}$$

Since not all new coins are handed in for re-coinage, households must weakly prefer not to hand in new coins and hence $e_1 \bar{\chi} \geq (1 - \tau) e_T$. Thus, $1 - \tau \leq \bar{\chi}^T$. When only new coins are held, appreciation is bounded above by $1/\bar{\chi}$, implying that $1 - \tau \geq \bar{\chi}^T$.

An implication of the theorem above is that there is a cutoff value for taxes that determines whether the Gesell tax generates revenues or not, i.e., whether coins are handed in for re-coinage or not. This level depends on market complexity, i.e., the number of goods in the economy. Let this cutoff be defined by

$$\hat{\tau}(K) = 1 - \bar{\chi}^T = 1 - \chi(\phi(\xi - g))^T \tag{39}$$

where ϕ depends on K through (18).

We now describe how a change in market complexity (in the sense of the number of goods in the economy) affects the system of taxation used by the lord. Note, from differentiating (18) in a cyclical equilibrium where c_t is constant, we have

$$\frac{d\phi}{dK} = \frac{w_{12}((1-\phi), K)}{-\frac{u'(\xi - g)}{\phi^2}(\xi - g) + \phi \bar{l} w_{11}((1-\phi), K)}. \tag{40}$$

The denominator is negative, since w is strictly concave. The numerator is negative from Lemma 1 and hence we have the following result.

Theorem 4 *If $K' > K$ then, in a cyclical equilibrium, $\phi' > \phi$ and $\hat{\tau}(K') > \hat{\tau}(K)$.*

Thus, if the economy is more advanced in the sense that there is a larger number of goods in the economy, bartering is more costly and hence households relies more on market transactions and the degree of monetization is higher. Then the probability of being discovered using illegal coins is smaller implying that the bound $(\chi(\phi(\xi - g)))^T$ on tax rates in Theorem 3 decreases. Thus the set of tax rates supporting positive revenues from the Gesell tax is smaller.

4.3 Relationship to empirical evidence

The empirical evidence in section 3.3 indicates that new coins almost exclusively circulated in England during a period when withdrawals occurred relatively infrequently (973–1035). After 1035, the intervals became shorter, which tightened the cutoff $\hat{\tau}$ in expression (39), and if the fee was unchanged, the shorter intervals also increased the implied yearly fee. When fees increase, old coins tend to be found much more frequently in hoards, which indicates that both old and new coins circulated together. Before 1035, hoards that contain only the last issue dominate - 83 percent of the hoards have only the last type—whereas after 1035, 33 percent of the hoards contain only the last type; see Svensson (2016), table 2. Regarding the number of coins from different issues in the hoards, the pattern is similar. Before 1035, the share of the last type is 86.5 percent, and after 1035, the share drops to 54.3 percent. Similar evidence from Thuringia in Germany, where the tax was 25 percent and withdrawals occurred every year, the coin hoards usually contain several types; see Svensson (2016), table 3. The share of hoards that contain only the last type is 2.4 percent, whereas the vast majority - more than 80 percent - contains three types or more.

Regarding prices, the evidence is scarce. However, some evidence of price regulation from the Frankish empire in the late 8th century seems to indicate that prices rose during a cycle, which is consistent with Theorem 3 (see also section 3.2).

Empirical observations show that periodic re-coinage broke down in England in the beginning of the 12th century and in Germany in the end of the 13 the century. The main reason was that the economies became more complex with more goods traded in the market and larger volumes of coins in circulation. Then, it was more difficult for the lords to monitor the short-lived coinage system. Thus, periodic re-coinage was replaced by a system with long-lived coins; see Svensson (2016).

5 Conclusions

A frequent method for generating revenues from seigniorage in the Middle Ages was to use Gesell taxes through periodic re-coinage. Under re-coinage, coins are legal tender only for a limited period of time. In such a short-lived coinage system, old coins are declared invalid and exchanged for new coins at publicly announced dates and exchange fees, which is similar to Gesell taxes. Empirical evidence based on several methods shows that re-coinage could occur as often as twice per year in a currency area during the Middle Ages. In contrast, in a long-lived coinage system, coins did not have a fixed period as the legal means of payment. Long-lived coins were common in western and southern Europe in the High Middle Ages, whereas short-lived coins dominated in central, northern and eastern Europe. Although the short-lived coinage system defined legal tender for almost 200 years in large parts of medieval Europe, it has seldom if ever been mentioned or analyzed in the literature of economics.

The main purpose of this study is to discuss the evidence for and analyze the consequences of short-lived coinage systems. In a short-lived coinage system, only one coin type may circulate in the currency area, and different coin types that reflect various issues must be clearly distinguishable for everyday users of the coins. The coin-issuing authority had several methods to monitor and enforce a re-coinage. First, there were exchangers and other administrators in the city markets. Second, the payment of any fees, taxes, rents, tithes or fines had to be made with the new coins. Although only new coins were allowed to be used for transactions, the evidence from coin hoards indicates that agents often also used illegal coins.

A cash-in-advance model is formulated to capture the implications of this monetary institution and it's relationship to the degree of complexity of the economy. The model includes households, firms and a lord, where households care about goods and jewelry consumption, and the firms care about profits. We capture complexity of the economy in terms of the number of goods that are traded. Bartering is costly in the sense that each bilateral meeting between traders carries a cost in terms of resources. When trading, households face a cash-in-advance constraint. Households can hold both new and old coins so that the equilibrium choice of which coins to hold is endogenous. The lord uses seigniorage to finance consumption.

A key results is that the system with Gesell taxes works, in the sense that agents participate in re-minting coins and the system generates tax revenues, the less complex the economy is. The reason is that the share of goods being traded on the market is smaller, since it is easier to find a double coincidence of wants when bartering. This, in turn, increases the probability that illegal coins are detected when the economy is less complex and has a lower degree of monetization. Furthermore, the system with Gesell taxes also works 1) if the tax is sufficiently low, 2) if the time period between two instances of re-coinage is sufficiently long and 3) if the probability of being penalized for using old illegal coins is sufficiently high. Also, prices increase over time during an issue period and fall immediately after the re-coinage date. Moreover, the higher the Gesell tax is, the higher the price increases are (as long as the coins are surrendered for re-coinage). Additionally, although nominal returns become negative when the Gesell tax is levied, real returns are unchanged because the price level adjusts accordingly as a result of the reduction in money holdings.

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A Appendix

A.1 Household optimization

Using the first-order conditions with respect to c_t and m^n_{t+1}, if $m^o_{t+1} > 0$ then

$$\beta \left(\frac{e_{t+1}\chi\left(\frac{m_{t+1}}{p_{t+1}}\right)}{e_t} - 1 \right) \frac{u'(c_{t+1})}{\phi_{t+1} p_t} \geq 0. \tag{A.1}$$

and if $m^n_{t+1} > 0$ then

$$\beta \left(\left((1-\mathbb{I}_t) + \mathbb{I}_t q^r_t\right) \frac{e_{t+1}\chi\left(\frac{m_{t+1}}{p_{t+1}}\right)}{e_t} - 1 \right) \frac{u'(c_{t+1})}{\phi_{t+1} p_t} \leq 0. \tag{A.2}$$

Since the consumer now holds new coins in period $t+1$, the exchange rates in period t and $t+1$ have to give the consumer incentives to not only hold old coins. For this to be the case, the exchange rate increase cannot be too large and is bounded above by $\left((1-\mathbb{I}_t) + \mathbb{I}_t q^r_t\right) / \chi\left(\frac{m_{t+1}}{p_{t+1}}\right)$.

Furthermore, using the first-order condition with respect to m^n_{t+1} gives

$$\beta \max \left\{ \frac{u'(c_{t+1})}{\phi_{t+1} p_{t+1}}, \frac{e_{t+1}\chi\left(\frac{m_{t+1}}{p_{t+1}}\right)}{e_t} \left((1-\mathbb{I}_t) + \mathbb{I}_t q^r_t\right) \frac{u'(c_{t+1})}{\phi_{t+1} p_{t+1}} \right\} \leq \frac{u'(c_t)}{\phi_t p_t}. \tag{A.3}$$

The conditions hold with equality only if the cash in advance constraint does not bind.

A.2 Proofs

Proof of Lemma 1.

Consider (10). Note that $\lim_{N_r \to \infty} f_N(N_r) = 0$ and that

$$\begin{aligned} \frac{df_N(N_r)}{dK} &= (N_r - 1)\left(1 - \frac{1}{K^2}\right)^{N_r - 2} \frac{1}{K^2} 2 \frac{1}{K^3} - 2\left(1 - \frac{1}{K^2}\right)^{N_r - 1} \frac{1}{K^3} \\ &= 2\left(1 - \frac{1}{K^2}\right)^{N_r - 2} \left(\frac{1}{K}\right)^3 \left(\frac{N_r}{K^2} - 1\right) \end{aligned} \tag{A.4}$$

Hence, for each K, K' so that $K' > K$ there is some $\bar{N}_r$ such that

$$\begin{aligned} f_N(N_r, K') &< f_N(N_r, K) \text{ for } N_r < \bar{N}_r \\ f_N(N_r, K') &> f_N(N_r, K) \text{ for } N_r > \bar{N}_r \end{aligned} \tag{A.5}$$

Since probabilities sum to one and $\hat{w}$ is decreasing, we have

$$\sum_{N_r=1}^{\infty} \hat{w}\left((1-\phi_t)\,\bar{l}N_r\right) f_N(N_r, K') > \sum_{N_r=1}^{\infty} \hat{w}\left((1-\phi_t)\,\bar{l}N_r\right) f_N(N_r, K) \tag{A.6}$$

Thus, w is increasing in K.

Also, since $\hat{w}$ is decreasing and strictly concave, $\hat{w}' < 0$ and $\hat{w}'' < 0$. Hence, by the same argument as for w it follows that w_1 is decreasing in K. ■

Proof of Lemma 2:

Note that, when analyzing e.g. money holdings in a cycle, the period where the fee is levied is important. Thus, when comparing a time period t to a point in the cycle, the notation $\text{mod}(t)$ should be used, with $\text{mod}(t) \in \{1, \ldots, T\}$. However, instead of writing e.g. $\text{mod}(t) < T$, we write $t < T$ and so on.

Step 1. Showing that ϕ and c increases in tandem. Suppose that $c_t' > c_t$. Barter trade c_t^B is $(1-\phi_t)\,\xi$ and market trade c_t^M is $\phi_t\xi - g + \text{Im}_t$. Thus, noting that $c_t = c_t^B + c_t^M$ if $c_t' > c_t$ then $\text{Im}_t' > \text{Im}_t$. Also, since Im_t only affect market consumption, we have $c_t' = c_t + \Delta_t$ and $c_t^{M\prime} = c_t^M + \Delta_t$ for some Δ_t. Then

$$\phi_t' = \frac{c_t^{M\prime}}{c_t'} = \frac{c_t^M + \Delta_t}{c_t + \Delta_t}.$$

Since the above expression is increasing in Δ_t as long as $c_t^M < c_t$ we have $\phi_t' > \phi_t$.

Step 2. We now describe conditions on imports. We have, using (7),

$$\frac{u'(c_t)}{\phi_t} = \beta\frac{u'(c_{t+1})}{\phi_{t+1}} + \gamma v'(d_{t+1}) \tag{A.7}$$

In a cyclical equilibrium, imports over a cycle must sum to zero. Hence, in case $\text{Im}_t > 0$ there must be some t' for which $\text{Im}_{t'} < 0$. Consider a t (with $t < T$) such that $\text{Im}_t > 0$ and $\text{Im}_{t+1} \leq 0$. Then $\text{Im}_t > \text{Im}_{t+1}$ and $c_{t+1} < c_t$. First, suppose $d_{t+2} \geq d_{t+1}$. Then, rewriting the Euler equation, we have

$$\beta \frac{u'(c_{t+2})}{\phi_{t+2}} = \frac{u'(c_{t+1})}{\phi_{t+1}} - \gamma v'(d_{t+2})$$

Since $c_{t+1} < c_t$ and $d_{t+2} \geq d_{t+1}$ and thus $u'(c_{t+1}) > u'(c_t)$ and $v'(d_{t+2}) \leq v'(d_{t+1})$ we have $u'(c_{t+2}) > u'(c_{t+1})$ and hence $c_{t+2} < c_{t+1}$ implying that $\text{Im}_{t+2} < \text{Im}_{t+1} < 0$. Second, suppose $d_{t+2} < d_{t+1}$. Since $d_{t+1} = d_t + h_t$ where $h_t = b(\mu_t^n + \mu_t^o - n_t^n) - \gamma \text{Im}_t$ this requires $-bn_{t+1}^n - \gamma \text{Im}_{t+1} \leq 0$. Then, since this implies that $n_{t+1}^n > 0$ we have $q_{t+1} = \frac{1-\tau}{b}$. Also, using Lemma 3, the CIA constraint (12), and if $m_{t+1}^o > 0$ then $e_{t+1}\chi\left(\frac{m_t}{p_t}\right) \geq e_t$ and $m_{t+1}^o = \chi\left(\frac{m_t}{p_t}\right) m_t^o - \mu_t^o$,

$$\begin{aligned} p_t(\phi_t c_t + g) &= m_{t+1}^n + e_t \chi\left(\frac{m_t}{p_t}\right) m_t^o \qquad \text{(A.8)} \\ p_{t+1}\left(\phi_{t+1} c_{t+1} - \frac{b}{\gamma} n_{t+1}^n\right) &= m_{t+1}^n + e_{t+1} \chi\left(\frac{m_{t+1}}{p_{t+1}}\right) m_{t+1}^o \end{aligned}$$

If $\mu_{t+1}^o > 0$ then $\frac{e_{t+1}}{b} < q_{t+1}$. Hence, using that $e_{t+1}\chi\left(\frac{m_t}{p_t}\right) \geq e_t$ whenever $m_{t+1}^o > 0$ we have $\frac{e_t}{b} < \frac{1-\tau}{b} \leq q_t$ and hence $\mu_t^o = \chi\left(\frac{m_t}{p_t}\right) m_t^o$, a contradiction. Thus, either $m_t^o = m_{t+1}^o = 0$ or $m_{t+1}^o = \chi\left(\frac{m_t}{p_t}\right) m_t^o$ and $e_{t+1}\chi\left(\frac{m_t}{p_t}\right) \geq e_t$ with equality when $m_{t+1}^n > 0$.

Then, since $q_{t+1} = \frac{1-\tau}{b}$ and $p_s = \gamma q_s$ we have $p_t \geq p_{t+1}$. Since $-bn_{t+1}^n - \gamma \text{Im}_{t+1} \leq 0$ we have

$$p_{t+1}(\phi_{t+1} c_{t+1}) > p_{t+1}\left(\phi_{t+1} c_{t+1} - \frac{b}{\gamma} n_{t+1}^n\right) = p_t(\phi_t c_t + g) \qquad \text{(A.9)}$$

and hence, since $n_{t+1}^n > 0$, and using $c_{t+1} < c_t$

$$p_t < \frac{\phi_{t+1} c_{t+1}}{\phi_t c_t + g} p_{t+1} < p_{t+1} \qquad \text{(A.10)}$$

Then $q_t < \frac{1-\tau}{b}$, a contradiction.

Now suppose $t = T$. Consider period $t+1$. For the case when $d_{t+2} \geq d_{t+1}$ we can proceed as above. Hence, $\text{Im}_{t+2} < \text{Im}_{t+1} < 0$. When $d_{t+2} < d_{t+1}$ we have, by similar arguments as above, that $n_{t+1}^n > 0$ and $q_{t+1} = \frac{1-\tau}{b}$. Using the CIA constraint (12)

$$\begin{aligned} p_t(\phi_t c_t + g) &= \frac{1}{1-\tau} m_{t+1}^n + e_t \chi\left(\frac{m_t}{p_t}\right) m_{t+1}^o \qquad \text{(A.11)} \\ p_{t+1}\left(\phi_{t+1} c_{t+1} - \frac{b}{\gamma} n_{t+1}^n\right) &= m_{t+1}^n + e_{t+1} \chi\left(\frac{m_{t+1}}{p_{t+1}}\right) m_{t+1}^o \end{aligned}$$

Proceeding as above establishes that (noting that, when $h_t^{rh} < m_t^n + \Pi_t^n$, we have $e_{t+1}\chi\left(\frac{m_{t+1}}{p_{t+1}}\right) \geq (1-\tau)\, e_t$)

$$(1-\tau)\, p_t < \frac{\phi_{t+1} c_{t+1}}{\phi_t c_t + g} p_{t+1} < p_{t+1}. \tag{A.12}$$

Consider period t. Note that $c_{t+1} < c_t$ from above. Suppose that $d_{t+1} > d_t$. If $d_{t+1} > d_t$ then $b\left(\mu_t^n + \mu_t^o - n_t^n\right) - \gamma\,\mathrm{Im}_t > 0$ implying that, as long as $\mathrm{Im}_t > 0$, we have $\mu_t^n > 0$ or $\mu_t^o > 0$ and hence $q_t \geq \frac{1}{b}$ (noting that $e_t \geq 1$ whenever $m_{t+1}^o > 0$ which is required for $m_t^o > 0$ in a cyclical equilibrium) and, from (A.12) implying $q_{t+1} > \frac{1-\tau}{b}$, a contradiction. If $\mathrm{Im}_t < 0$ then, from step 1, there is some s such that $\mathrm{Im}_s > 0$ and $\mathrm{Im}_{s+1} < 0$. Then, repeating the arguments in step 1, we have $p_s < \frac{1}{1-\tau} p_{T+1}$. Unless $\mu_s^n > 0$ or $\mu_s^o > 0$ (which, since $q_s \geq \frac{1}{b}$, leads to a contradiction) we have $d_{s+1} < d_s$. Since $c_{s+1} < c_s$ we have, using the Euler equation, $c_s < c_{s-1}$. Again, unless $\mu_{s-1}^n > 0$ or $\mu_{s-1}^o > 0$ (which again leads to a contradiction) we have $d_s < d_{s-1}$. Repeating until $s-1 = 1$ establishes that $\mu_s^n = \mu_s^o = 0$ but $n_1^n > 0$, contradicting cyclicality.

Suppose $d_{t+1} \leq d_t$. Then, proceeding as above, $c_t < c_{t-1}$ and $\mathrm{Im}_t < \mathrm{Im}_{t-1}$. Repeatedly using the Euler equation as above, establishes $\mu_s^n = \mu_s^o = 0$ but $n_1^n > 0$, again contradicting cyclicality.

By induction, $\mathrm{Im}_{t+s} < 0$ for all $s > 0$, contradicting cyclicality.■

Proof of Lemma 3:

Case 1**.** First, suppose that $t \neq T$.

Suppose that $\mu_t^o = 0$. If $n_t^n > 0$ then $q_t = \frac{1-\tau}{b}$ from (4) and thus, $\mu_t^n = 0$. Using (2), (7), (21), that the Inada conditions imply that (12) holds with equality and, from (2) that $h_t = -b n_t^n - \gamma\,\mathrm{Im}_t$, we get

$$p_t\left(\phi_t c_t + g - \mathrm{Im}_t\right) = m_{t+1}^n + e_t \chi\left(\frac{m_t}{p_t}\right) m_t^o. \tag{A.13}$$

A similar argument holds if $n_t^n = \mu_t^n = 0$.

Suppose that $\mu_t^n > 0$ so that $q_t = \frac{1}{b}$ from (5). Then

$$p_t \phi_t c_t = m_t^n + e_t \chi\left(\frac{m_t}{p_t}\right) m_t^o. \tag{A.14}$$

Using $h_t = b\mu_t^n - \gamma\,\mathrm{Im}_t$ and money transition (21) we get $p_t\left(c_t + g - \mathrm{Im}_t\right) = m_{t+1}^n + e_t \chi m_t^o$.

A similar argument holds if $\mu_t^o > 0$. We get

$$p_t(\phi_t c_t + g - \mathrm{Im}_t) = m_{t+1}^n + e_t m_{t+1}^o. \tag{A.15}$$

Case 2**.** Now, suppose that $t = T$.

Suppose that $\mu_t^o = 0$. If $n_t^n > 0$**,** we can proceed as in Case 1 to establish

$$p_t \phi_t c_t + q_t h_t = m_t^n + e_t \chi\left(\frac{m_t}{p_t}\right) m_t^o. \tag{A.16}$$

We have $m_{t+1}^n = (1-\tau) h_t^r$ and

$$h_t^r \in [0, m_t^n + p_t g + (1-\tau) n_t^n - \mu_t^n] \tag{A.17}$$

If h_t^{rh} is equal to the upper bound, we can proceed as above to establish $p_t(\phi_t c_t + g - \mathrm{Im}_t) = \frac{1}{1-\tau} m_{t+1}^n$. If $h_t^{rh} < 1$ then $e_T \geq 1$ from (17) and thus, using (21), we have, using the constraints imposed on q_t and e_t when minting or melting is positive gives

$$\begin{aligned} p_t(\phi_t c_t + g - \mathrm{Im}_t) &= \frac{1}{1-\tau} e_t m_{t+1}^n + (1-e_t)(m_t^n + p_t g) \\ &\quad - (e_t - 1)(1-\tau) n_t^n + (e_t - 1) \mu_t^n + e_t m_{t+1}^o \end{aligned} \tag{A.18}$$

A similar argument holds if $\mu_t^n > 0$, if $\mu_t^o > 0$ and if $\mu_t^n = n_t^n = 0$. If h_t^{rh} is interior then $e_T = 1$ implying that $p_t(\phi_t c_t + g - \mathrm{Im}_t) = \frac{1}{1-\tau} m_{t+1}^n + e_t m_{t+1}^o$ ■

Proof of Lemma 4

Define $\bar{\chi}_t = \chi\left(\frac{m_t}{p_t}\right)$. We prove Lemma 4 by contradiction. Suppose that $n_s^n > 0$ for some $1 \leq s \leq T-1$.

Step 1**.** Finding a relationship between current and tomorrows money holdings and showing $\mu_s^o = 0$ whenever $m_s^o > 0$ and $s < T$.

Since $n_s^n > 0$ we have, from (4) that $q_s \leq \frac{1-\tau}{b}$. From (22), $m_s^o > 0$ requires $h_T^{rh} < \Pi_t^n$ for cyclicality to be satisfied and hence, from the household optimality condition for h_T^{rh}, we have that $e_T \geq 1$. Also, using that $m_r^o > 0$ for $r < s$ and (8) and (15) we have $e_1 \bar{\chi}_1 \geq (1-\tau) e_T \geq (1-\tau)$ and $e_r \bar{\chi}_r \geq e_{r-1}$ so that $e_r > 1-\tau$ for $r \leq s$ and hence $\frac{e_s}{b} > q_s$. Then, using the optimality condition for melting old coins, we have $\mu_s^o = 0$ and

$m^o_{s+1} = \bar{\chi}_s m^o_s$. Then, $e_s m^o_{s+1} = e_s \bar{\chi}_s m^o_s \geq e_{s-1} m^o_s$. Using (21),

$$m^n_{s+1} + e_s \bar{\chi}_s m^o_s = m^n_s + p_s g + e_s m^o_{s+1} + (1-\tau)\, n^n_s > m^n_s + p_s g + e_{s-1} m^o_s. \tag{A.19}$$

Then, using Lemma 2 and 3, letting $c^M = \phi(\xi - g)$, we have

$$m^n_{s+1} + e_s m^o_{s+1} > \frac{\phi(\xi-g)}{\phi(\xi-g)-g}\left(m^n_s + e_{s-1} m^o_s\right) \tag{A.20}$$

so that, using Lemma 3, we have $p_s > \frac{\phi(\xi-g)}{\phi(\xi-g)-g} p_{s-1}$ and hence $Q_s > \frac{\phi(\xi-g)}{\phi(\xi-g)-g}$.

Finally, since $q_{s-1} \geq \frac{1-\tau}{b}$ and $q_{s+1} \geq \frac{1-\tau}{b}$ we have, using (32), that $n^n_s > 0$ implies $d_{s+1} < d_s$ and the concavity of v, that $Q_{s+1} = \frac{p_{s+1}}{p_s} > Q_s$, implying that $p_{s+1} > \frac{\phi(\xi-g)}{\phi(\xi-g)-g} p_s$.

Step 2. Showing that $n^n_{s+1} > 0$.

Case 1. $s+1 \leq T$. From step 1,

$$p_{s+1}\phi(\xi-g) > p_s \phi(\xi-g) + p_{s+1} g \tag{A.21}$$

Then, using Lemma 3 for p_s and (12) for p_{s+1}, we have

$$m^n_{s+1} + e_{s+1}\bar{\chi} m^o_{s+1} - q_{s+1} h_{s+1} > m^n_{s+1} + e_s \bar{\chi}_s m^o_s + (p_{s+1} - p_s)\, g. \tag{A.22}$$

Since $m^n_{s+1} > 0$ we have $e_{s+1}\bar{\chi}_{s+1} \leq e_s$ and $m^o_{s+1} \leq \bar{\chi}_s m^o_s$ and hence $-q_{s+1} h_{s+1} > 0$, implying that $n^n_{s+1} > 0$.

Case 2. $s+1 = T+1$. Using Lemma 3 for p_s and (12) for p_{s+1} gives

$$\begin{aligned} m^n_{s+1} + e_{s+1}\bar{\chi}_{s+1} m^o_{s+1} - q_{s+1} h_{s+1} \;&>\; \frac{1}{1-\tau} e_s m^n_{s+1} + e_s m^o_{s+1} + (p_{s+1} - p_s)\, g \\ &+ (1-e_s)\left(m^n_{s+1} + p_{s+1} g + (1-\tau)\, n^n_{s+1} - \mu^n_{s+1}\right). \end{aligned} \tag{A.23}$$

Hence , if $m^n_{s+1} > 0$ so that $h^{rh}_s > 0$ then $e_s = 1$ and we get

$$m^n_{s+1} + e_{s+1}\bar{\chi}_{s+1} m^o_{s+1} - q_{s+1} h_{s+1} > \frac{1}{1-\tau} m^n_{s+1} + e_s m^o_{s+1} + (p_{s+1} - p_s)\, g. \tag{A.24}$$

Then, using $e_{s+1}\chi_{s+1} \leq (1-\tau)\, e_s$ and proceeding as above establishes that $-q_{s+1} h_{s+1} > 0$.

If $m^n_{s+1} = 0$ then, using (12) for p_s and p_{s+1},

$$e_{s+1}\bar{\chi}_{s+1}m^o_{s+1} - q_{s+1}h_{s+1} > m^n_s + e_s\bar{\chi}_s m^o_s - q_s h_s + p_{s+1}g \tag{A.25}$$

If $n^n_{s+1} = 0$ then $h_{s+1} \geq 0$. If $e_{s+1} \leq 1$ then, using (15) we have $e_{s+1}\bar{\chi} \leq 1$ and we have

$$m^o_{s+1} > m^n_s + \bar{\chi}_s m^o_s + p_{s+1}g + (1-\tau)\left((1-\tau)\, n^n_s - \mu^n_s\right) \tag{A.26}$$

a contradiction. Hence, $e_{s+1} > 1$. Suppose $\mu^o_{\hat{t}} > 0$ for some $\hat{t}$ (and $\mu^o_t = 0$ for $t < \hat{t}$). Then, for all $t \leq \hat{t}$, using (15), we have $e_t\bar{\chi}_t \geq e_{t-1}$ and thus $e_t > 1$. Since $\mu^o_{\hat{t}} > 0$ we have $\frac{1}{b} < \frac{e_{\hat{t}}}{b} \leq q_{\hat{t}}$, implying that $\mu^n_t = \infty$ from the optimality condition for melting new coins, a contradiction. Suppose $\mu^o_t = 0$ for all t and $\mu^n_{\hat{t}} > 0$ for some $\hat{t}$ (and $\mu^n_t = 0$ for $t > \hat{t}$. This follows since by case 1 above we cannot have $n^n_r > 0$ for $r \leq \hat{t}$ since then $n^n_s > 0$ for $s \in \{\hat{t}+1, t'\}$). Then $q_{\hat{t}} = \frac{1}{b}$ and there is some $t' > \hat{t}$ such that $q_{t'} = \frac{1-\tau}{b}$ implying that $p_{t'} < p_{\hat{t}}$. However, from above we have $Q_s \geq 1$ for $s \in \{\hat{t}+1, t'\}$ a contradiction.

Step 4. Induction.

By induction we have $n^n_t > 0$ for all $t \geq 1$, contradicting cyclicality.■

Proof of Theorem 3.

As in Lemma 4, we denote the constant value χ over the cycle as $\bar{\chi}$.

From Lemma 2, $n^n_t = 0$, $\mu^n_t = 0$ and $\mu^o_t = 0$ for all t.

Preliminaries. From money transition (21), we have, except when $t = T$, using Lemma 3,

$$m^n_{t+1}\frac{\phi c}{\phi c + g} = m^n_t + e_t\bar{\chi}m^o_t\frac{g}{\phi c + g}. \tag{A.27}$$

Using Lemma 3 gives

$$\frac{p_t}{p_{t-1}} = \frac{m^n_{t+1} + e_t\bar{\chi}m^o_t}{m^n_t + e_{t-1}\bar{\chi}m^o_{t-1}} = \frac{\phi c + g}{\phi c} + \frac{\phi c + g}{\phi c}\frac{e_t\bar{\chi}m^o_t - e_{t-1}\bar{\chi}m^o_{t-1}}{m^n_t + e_{t-1}\bar{\chi}m^o_{t-1}}. \tag{A.28}$$

If $m^o_t > 0$ and $m^n_t > 0$, then, using money transition (22) and, from (15), that we have $e_t\bar{\chi} = e_{t-1}$ and $m^o_t = \bar{\chi}m^o_{t-1}$, the last term in (A.28) is zero. If $m^o_t = 0$ then, since $\mu^o_{t-1} = 0$ we have $m^o_{t-1} = 0$ again the last term is zero. Thus,

$$\frac{p_{t-1}}{p_t} = \frac{\phi c}{\phi c + g}. \tag{A.29}$$

Case 1. $m^o_{t+1} = 0$ for all t.

Step 1. Since $h^{rh}_T = m^n_T$ we have, from (16) and the household optimality condition for h^{rh}_T, that $q^r_T \geq \frac{1}{1-\tau}$, $e_1\bar{\chi} \leq \frac{e_T}{q^r_T}$, $e_{t+1}\bar{\chi} \leq e_t$ and $1 \geq e_T$ and hence

$$\frac{e_T}{q^r_T} \geq e_1\bar{\chi} \geq e_2\bar{\chi}^2 \geq \ldots \geq e_T\bar{\chi}^T \iff 1-\tau \geq \bar{\chi}^T. \tag{A.30}$$

Step 2. Prices.

We have, using Lemma 3, (21) and that (A.29) holds, for $t \neq T$,

$$\frac{\phi c}{\phi c + g} m^n_{t+1} = m^n_t \tag{A.31}$$

and, using (19),

$$\tau h^{rh}_T = \sum_{t=1}^{T} p_t g = \sum_{t=1}^{T} m^n_{t+1} \frac{g}{\phi c} = \frac{g}{\phi c} \sum_{t=1}^{T} \left(\frac{\phi c}{\phi c + g}\right)^{T-t} m^n_T \tag{A.32}$$

so that, using $h^{rh}_T = m^n_T + p_T g = \frac{1}{1-\tau} m^n_1$ and $m^n_T = m^n_1 - p_T g = \frac{\phi c}{\phi c+g}\frac{1}{1-\tau} m^n_1$,

$$\tau \frac{\phi c + g}{\phi c} = \frac{g}{\phi c} \sum_{t=2}^{T+1} \left(\frac{\phi c}{\phi c + g}\right)^{T-t+1} = \frac{\phi c + g}{\phi c}\left(1 - \left(\frac{\phi c}{\phi c + g}\right)^T\right) \tag{A.33}$$

and hence $\frac{\phi c}{\phi c+g} = (1-\tau)^{\frac{1}{T}}$ so that $\phi c = (1-\tau)^{\frac{1}{T}} (\phi(\xi - g) + g)$. From (A.31), for $t = 2, \ldots, T$, we have $(1-\tau)^{\frac{1}{T}} p_t = p_{t-1}$ and thus $p_1 = (1-\tau)^{\frac{T-1}{T}} p_T$. Since $q_T \leq \frac{1}{b}$ from the optimality condition for melting new coins, any $q_1 \in [\frac{1-\tau}{b}, \frac{(1-\tau)^{\frac{T-1}{T}}}{b}]$ is possible, implying that $q_T \in [\frac{(1-\tau)^{\frac{1}{T}}}{b}, \frac{1}{b}]$.

Using that $c = \frac{(1-\tau)^{\frac{1}{T}}}{\phi + (1-\tau)^{\frac{1}{T}}(1-\phi)}\xi$ in (18) gives a solution for ϕ. Then, from $(1-\tau)^{\frac{1}{T}} p_t = p_{t-1}$, the Cash in Advance constraint $p_T\phi c = \frac{1}{1-\tau} m^n_1$ and $p_T = \gamma q_T$, for each $q_T \in [\frac{(1-\tau)^{\frac{1}{T}}}{b}, \frac{1}{b}]$, there is a unique m^n_1 that satisfies the Cash in Advance constraint. Furthermore, we have $\frac{dp_T}{dm^n_1} > 0$.

Step 3. Finding m^n_1.

Using that $c = \frac{(1-\tau)^{\frac{1}{T}}}{\phi + (1-\tau)^{\frac{1}{T}}(1-\phi)}\xi$, $\frac{q_1}{p_1} = \frac{q_T}{p_T} = \frac{1}{\gamma}$, and silver market clearing $d_1 = \ldots = d_T = S - b\left(m^n_1 + m^L_1\right)$, equation (14) is

$$v'^{-1}\left(u'\left(\frac{(1-\tau)^{\frac{1}{T}}}{\phi + (1-\tau)^{\frac{1}{T}}(1-\phi)}\xi\right)\frac{1}{\gamma\phi}(1-\beta)\right) = S - b\left(m^n_1 + m^L_1\right). \tag{A.34}$$

Then, using that $p_T\phi c = \frac{1}{1-\tau}m_1^n$ and that ϕ is determined by (18), for each $q_T \in [\frac{(1-\tau)^{\frac{1}{T}}}{b}, \frac{1}{b}]$, there is a unique S that satisfies the Euler equation. Furthermore, by differentiating the Euler equation, we have $\frac{dp_T}{dS} > 0$.

Case 2. $m_{t+1}^o > 0$ for all t.

Step 1. Exchange rates.

Using that $\mu_t^o = 0$ from Lemma 2 and, since $\mu_t^n = 0$ implies $m_t^n > 0$ for $t \neq 1$, that $e_t\bar{\chi} = e_{t-1}$ from (15) and (16) and, using from the household optimality condition for h_T^{rh}, $e_T \geq 1$, we have $e_t \geq \bar{\chi}^{T-t}$. Moreover, if $h_T^{rh} \in (0,1)$ then $e_T = 1$ and $q_T^r e_{T+1}\bar{\chi} = e_T$. Combining this and $e_t = \bar{\chi}^{T-t}$ establishes that $\bar{\chi}^T = 1 - \tau$ whenever $h_T^{rh} \in (0,1)$.

Step 2. Showing $\bar{\chi} \leq \frac{\phi(\xi-g)}{\phi(\xi-g)+g}$.

Since $\mu_t^o = 0$ for all t, we have $m_t^o = \bar{\chi}m_{t-1}^o$. Then, using (22) we have $m_1^o = \chi m_T^o + \left(m_T^n + p_T g - h_T^{rh}\right)$ and $m_t^o = \bar{\chi}m_{t-1}^o$ and hence $m_1^o = \frac{1}{1-\chi^T}\left(m_T^n + p_T g - h_T^{rh}\right)$ and, by repeatedly using $m_t^o = \bar{\chi}m_{t-1}^o$,

$$m_{t+1}^o = \frac{\bar{\chi}^t}{1-\bar{\chi}^T}\left(m_T^n + p_T g - h_T^{rh}\right). \tag{A.35}$$

Government revenues during a cycle are, in terms of new coins, using (A.35),

$$\tau h_T^{rh} + (1-\bar{\chi})\sum_{t=1}^{T} m_t^o = \tau h_T^{rh} + m_T^n + p_T g - h_T^{rh}. \tag{A.36}$$

To find government expenditures, using Lemma 3, that $m_t^n > 0$ for $t \neq 1$ since $\mu_t^n = 0$ and new coin dividends are positive, that $e_{t-1} = \bar{\chi}e_t$ and $m_t^o = \bar{\chi}m_{t-1}^o$ from (15), (16) and (22), we can write $p_t\phi c = m_t^n + e_1\bar{\chi}m_1^o$. Then

$$\sum_{t=1}^{T} p_t g = \frac{g}{\phi c + g}\frac{\phi c + g}{\phi c}\left(\sum_{t=1}^{T} m_t^n + Te_1\bar{\chi}m_1^o\right). \tag{A.37}$$

Using money transition (21) when $t \neq 1$,

$$m_t^n = \frac{\phi c}{\phi c + g}m_{t+1}^n - \frac{g}{\phi c + g}e_1\bar{\chi}m_1^o. \tag{A.38}$$

Solving the above expression for m_t^n and repeatedly substituting gives

$$m_t^n = \left(\frac{\phi c}{\phi c + g}\right)^{T-t} m_T^n - e_1 \bar{\chi} m_1^o \left(1 - \left(\frac{\phi c}{\phi c + g}\right)^{T-t}\right). \tag{A.39}$$

Then, summing and equating expenditures with revenues, using (A.36) and (A.37), we get

$$\tau h_T^{rh} + m_T^n + p_T g - h_T^{rh} = \left(1 - \left(\frac{\phi c}{\phi c + g}\right)^{T}\right)\left(m_T^n + p_T g + e_T \frac{\bar{\chi}^T}{1 - \bar{\chi}^T}\left(m_T^n + p_T g - h_T^{rh}\right)\right). \tag{A.40}$$

This implies

$$\left(1 - \left(\frac{\phi c}{\phi c + g}\right)^{T}\right)\left(\frac{1 - \bar{\chi}^T (1 - e_T)}{1 - \bar{\chi}^T}\right) = 1 \tag{A.41}$$

Suppose that $h_T^{rh} > 0$. Then, using (17), $e_T = 1$ and, using (15) and (16), $e_{t-1} = \bar{\chi} e_t$. Moreover, from (17) and (A.35), $e_T = 1$ so that $1 - \tau = \bar{\chi}^T$. and hence $1 - \bar{\chi}^T = 1 - \left(\frac{\phi c}{\phi c + g}\right)^T$ so that $\bar{\chi} = \frac{\phi c}{\phi c + g}$ and hence

$$c = \frac{\bar{\chi}}{\phi + (1 - \phi)\bar{\chi}} \xi \tag{A.42}$$

Suppose $h_T^{rh} = 0$ so that $e_T \geq 1$. Letting $\tau^* = \frac{1 - \bar{\chi}^T}{1 - \bar{\chi}^T (1 - e_T)}$ we have $\frac{\phi c}{\phi c + g} = (1 - \tau^*)^{\frac{1}{T}}$ and we can proceed as in Case 1 and thus

$$c = \frac{(1 - \tau^*)^{\frac{1}{T}}}{\phi + (1 - \tau^*)^{\frac{1}{T}} (1 - \phi)} \xi. \tag{A.43}$$

Note, however, that consumption is weakly larger than the right-hand side of (A.42).

Step 3. Prices.

From we have $\frac{\phi c}{\phi c + g} p_t = p_{t-1}$, and hence, $p_1 = \left(\frac{\phi c}{\phi c + g}\right)^T p_T$. Since, using the optimality condition for melting new coins, $q_T \leq \frac{1}{b}$ any $q_1 \in [\frac{1-\tau}{b}, \frac{\frac{\phi c}{\phi c + g}^T}{b}]$ is possible.

Step 4. Finding m_1^n.

Fix h_T^{rh}. Using (A.35) and that $m_t^o = \bar{\chi} m_{t-1}^o$, we can write, letting $s_T^n = \frac{h_T^{rh}}{m_1^n}$,

$$p_T = Z\left(\bar{\chi}, s_T^n, \phi c\right) m_T^n. \tag{A.44}$$

where

$$Z\left(\bar{\chi}, s_T^n, \xi\right) = \frac{1 + \frac{e_T \bar{\chi}^T}{1-\bar{\chi}^T}\left(1 - s_T^n\right)}{\phi c - \frac{e_T \bar{\chi}^T}{1-\bar{\chi}^T} g}. \tag{A.45}$$

Moreover, using (A.35), we have, letting

$$B\left(s_T^n\right) = 1 + \frac{\bar{\chi}^{T-1}}{1-\bar{\chi}^T}\left(1 + Z\left(\bar{\chi}, s_T^n, \phi c\right) g - s_T^n\right), \tag{A.46}$$

that $m_T^n + m_T^o = B\left(s_T^n\right) m_T^n$. Using that c is determined from (A.42) or (A.43), $\frac{q_1}{p_1} = \frac{q_T}{p_T} = \frac{1}{\gamma}$ and the silver market clearing condition $d_1 = \ldots = d_T = S - b\left[B m_T^n + m_T^L\right]$ we can, using (A.35) and (7), write the Euler equation (14) as,

$$\gamma = \frac{1}{1-\beta}\frac{\phi}{u'(c)} v'\left(S - b\left[B\left(s_T^n\right) m_T^n + m_T^L\right]\right). \tag{A.47}$$

The right-hand side is continuous and increasing in m_T^n due to concavity of v. Then, for each $q_T \in \left[\frac{(1-\tau)\bar{\chi}^{-T}}{b}, \frac{1}{b}\right]$ implying a unique p_T and, in turn, m_T^n there is a unique S that satisfies the Euler equation. Furthermore, by differentiating the Euler equation, we have $\frac{dp_T}{dS} > 0$. ■

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Two-Sided Network Effects, Bank Interchange Fees, and the Allocation of Fixed Costs *by Mats A. Bergman*	2005:185
Trade Deficits in the Baltic States: How Long Will the Party Last? *by Rudolfs Bems and Kristian Jönsson*	2005:186
Real Exchange Rate and Consumption Fluctuations follwing Trade Liberalization *by Kristian Jönsson*	2005:187
Modern Forecasting Models in Action: Improving Macroeconomic Analyses at Central Banks *by Malin Adolfson, Michael K. Andersson, Jesper Lindé, Mattias Villani and Anders Vredin*	2005:188
Bayesian Inference of General Linear Restrictions on the Cointegration Space *by Mattias Villani*	2005:189
Forecasting Performance of an Open Economy Dynamic Stochastic General Equilibrium Model *by Malin Adolfson, Stefan Laséen, Jesper Lindé and Mattias Villani*	2005:190
Forecast Combination and Model Averaging using Predictive Measures *by Jana Eklund and Sune Karlsson*	2005:191
Swedish Intervention and the Krona Float, 1993-2002 *by Owen F. Humpage and Javiera Ragnartz*	2006:192
A Simultaneous Model of the Swedish Krona, the US Dollar and the Euro *by Hans Lindblad and Peter Sellin*	2006:193
Testing Theories of Job Creation: Does Supply Create Its Own Demand? *by Mikael Carlsson, Stefan Eriksson and Nils Gottfries*	2006:194
Down or Out: Assessing The Welfare Costs of Household Investment Mistakes *by Laurent E. Calvet, John Y. Campbell and Paolo Sodini*	2006:195
Efficient Bayesian Inference for Multiple Change-Point and Mixture Innovation Models *by Paolo Giordani and Robert Kohn*	2006:196
Derivation and Estimation of a New Keynesian Phillips Curve in a Small Open Economy *by Karolina Holmberg*	2006:197
Technology Shocks and the Labour-Input Response: Evidence from Firm-Level Data *by Mikael Carlsson and Jon Smedsaas*	2006:198
Monetary Policy and Staggered Wage Bargaining when Prices are Sticky *by Mikael Carlsson and Andreas Westermark*	2006:199
The Swedish External Position and the Krona *by Philip R. Lane*	2006:200

Price Setting Transactions and the Role of Denominating Currency in FX Markets 2007:201
by Richard Friberg and Fredrik Wilander

The geography of asset holdings: Evidence from Sweden 2007:202
by Nicolas Coeurdacier and Philippe Martin

Evaluating An Estimated New Keynesian Small Open Economy Model 2007:203
by Malin Adolfson, Stefan Laséen, Jesper Lindé and Mattias Villani

The Use of Cash and the Size of the Shadow Economy in Sweden 2007:204
by Gabriela Guibourg and Björn Segendorf

Bank supervision Russian style: Evidence of conflicts between micro- and macro-prudential concerns 2007:205
by Sophie Claeys and Koen Schoors

Optimal Monetary Policy under Downward Nominal Wage Rigidity 2007:206
by Mikael Carlsson and Andreas Westermark

Financial Structure, Managerial Compensation and Monitoring 2007:207
by Vittoria Cerasi and Sonja Daltung

Financial Frictions, Investment and Tobin's q 2007:208
by Guido Lorenzoni and Karl Walentin

Sticky Information vs Sticky Prices: A Horse Race in a DSGE Framework 2007:209
by Mathias Trabandt

Acquisition versus greenfield: The impact of the mode of foreign bank entry on information and bank lending rates 2007:210
by Sophie Claeys and Christa Hainz

Nonparametric Regression Density Estimation Using Smoothly Varying Normal Mixtures 2007:211
by Mattias Villani, Robert Kohn and Paolo Giordani

The Costs of Paying – Private and Social Costs of Cash and Card 2007:212
by Mats Bergman, Gabriella Guibourg and Björn Segendorf

Using a New Open Economy Macroeconomics model to make real nominal exchange rate forecasts 2007:213
by Peter Sellin

Introducing Financial Frictions and Unemployment into a Small Open Economy Model 2007:214
by Lawrence J. Christiano, Mathias Trabandt and Karl Walentin

Earnings Inequality and the Equity Premium 2007:215
by Karl Walentin

Bayesian forecast combination for VAR models 2007:216
by Michael K. Andersson and Sune Karlsson

Do Central Banks React to House Prices? 2007:217
by Daria Finocchiaro and Virginia Queijo von Heideken

The Riksbank's Forecasting Performance 2007:218
by Michael K. Andersson, Gustav Karlsson and Josef Svensson

Macroeconomic Impact on Expected Default Freqency 2008:219
by Per Åsberg and Hovick Shahnazarian

Monetary Policy Regimes and the Volatility of Long-Term Interest Rates 2008:220
by Virginia Queijo von Heideken

Governing the Governors: A Clinical Study of Central Banks 2008:221
by Lars Frisell, Kasper Roszbach and Giancarlo Spagnolo

The Monetary Policy Decision-Making Process and the Term Structure of Interest Rates 2008:222
by Hans Dillén

How Important are Financial Frictions in the U S and the Euro Area 2008:223
by Virginia Queijo von Heideken

Block Kalman filtering for large-scale DSGE models 2008:224
by Ingvar Strid and Karl Walentin

Optimal Monetary Policy in an Operational Medium-Sized DSGE Model 2008:225
by Malin Adolfson, Stefan Laséen, Jesper Lindé and Lars E. O. Svensson

Firm Default and Aggregate Fluctuations 2008:226
by Tor Jacobson, Rikard Kindell, Jesper Lindé and Kasper Roszbach

Sveriges Riksbank
Visiting address: Brunkebergs torg 11
Mail address: se-103 37 Stockholm

Website: www.riksbank.se
Telephone: +46 8 787 00 00, Fax: +46 8 21 05 31
E-mail: registratorn@riksbank.se